24 Aug 2020

MA 598 Sect 528 AANT

Analytic Number Theory : a first course

§1. Introduction: generating functions.

Generating functions : $A(z) = \sum_{l=0}^{\infty} a_l z^l$ (formal)

$(a_l \in \mathbb{C})$

If $A(z)$ converges appropriately, then

$$a_n = \frac{1}{2\pi i} \oint \frac{A(z)}{z^{n+1}}\, dz .$$

If $B(z) = \sum_{m=0}^{\infty} b_m z^m$ is a 2nd such power series,

then

$$A(z)B(z) = \sum_{l=0}^{\infty} \sum_{m=0}^{\infty} a_l b_m z^{l+m} = \sum_{n=0}^{\infty} c_n z^n ,$$

where

$$c_n = \sum_{l=0}^{n} \sum_{\substack{m=0 \\ l+m=n}}^{n} a_l b_m \quad (n \geq 0)$$

② Power series $\longleftrightarrow$ additive structure

Fourier series: $z = e^{2\pi i\alpha}$ $\left(=: e(\alpha) \right)$

$$\tilde{A}(\alpha) = \sum_{l=0}^{\infty} a_l \, e(l\alpha)$$

Multiplicative structure? Generating functions

Dirichlet Series: $\alpha(s) = \sum_{l=1}^{\infty} a_l \, l^{-s}$. (formal)

Moreover, if $\beta(s) = \sum_{m=1}^{\infty} b_m \, m^{-s}$, then

$$\alpha(s)\,\beta(s) = \sum_{l=1}^{\infty} \sum_{m=1}^{\infty} a_l b_m \, (lm)^{-s} = \sum_{n=1}^{\infty} c_n n^{-s},$$

where

$$c_n = \sum_{\substack{l=1 \\ lm=n}}^{n} \sum_{m=1}^{n} a_l b_m \quad (n \geq 1).$$

③ Dirichlet series $\longleftrightarrow$ multiplicative structure.

Fourier series viewpoint specialises $s = it$ $(t \in \mathbb{R})$

$$\tilde{\alpha}(it) = \sum_{l=1}^{\infty} a_l \, l^{-it} = \sum_{l=1}^{\infty} a_l \, e\left(-\frac{t}{2\pi}\log l\right)$$

(frequencies $\log l$ $(l \in \mathbb{N})$).

Convention: $\underset{\mathbb{C}}{\overset{s}{\cap}} = \sigma + it$ $(\sigma \in \mathbb{R}, \ t \in \mathbb{R})$.

Example 1.1: Central example — Riemann Zeta function

Formal
$$\zeta(s) = \sum_{n=1}^{\infty} n^{-s}. \qquad (1.1)$$

However, when $\sigma = \mathrm{Re}(s) > 1$, ~~then~~

④

$$\sum_{n=1}^{\infty} |n^{-s}| \leq \sum_{n=1}^{\infty} n^{-\sigma} < \infty$$, so absolutely convergent.

Gives def.^{n} of $\zeta(s)$ for $Re(s) > 1$, and can extended to an analytic continuation for $s \in \mathbb{C} \setminus \{1\}$.

Multiplicative structure of Dirichlet series has potential to reveal properties of coeffts: Ex 1·1.

When $Re(s) = \sigma > 1$, ~~then~~

$$\sum_{n=1}^{\infty} n^{-s} = \sum_{n=1}^{\infty} (p_1^{h_1} \cdots p_r^{h_r})^{-s} = \prod_{p} (1 + p^{-s} + p^{-2s} + \cdots)$$

$h = p_1^{h_1} \cdots p_r^{h_r}$ (prime factorisation)

p all distinct primes

Whence

$$\zeta(s) = \prod_{p} (1 - p^{-s})^{-1} \quad (\sigma > 1). \quad - (1{\cdot}2)$$

"Euler Product".

⑤ Motivated by this:

$$\sum_{n \le x} \frac{1}{n} = \log x + O(1)$$

$$\wedge\!\!\!\!\!\!\; \mathbin{/\!/}$$

$$\prod_{p \le x} \left(1 + \frac{1}{p} + \frac{1}{p^2} + \cdots\right) = \prod_{p \le x} \left(1 - 1/p\right)^{-1}$$

Thus

$$-\sum_{p \le x} \log\left(1 - \frac{1}{p}\right) \geq \log\log x + O\left(\frac{1}{\log x}\right)$$

$$\|$$

$$\sum_{p \le x} \left(\frac{1}{p} + \frac{1}{2p^2} + \cdots\right) = \sum_{p \le x} \frac{1}{p} + O(1)$$

Then $$\sum_{p \le x} \frac{1}{p} \geq \log\log x + O(1) \xrightarrow[x \to \infty]{} \infty$$

In particular, there are ∞ly many primes.

Standard notation

$$\pi(x) := \sum_{p \leq x} 1 .$$

Crudely, if $\pi(x) = n \in \mathbb{N}$, then

$$\sum_{m \leq n} \frac{1}{m} \geq \sum_{p \leq x} \frac{1}{p} \geq \log\log x + O(1)$$

$$\Longrightarrow \quad \log n \geq \log\log x + O(1) \longrightarrow \pi(x) \gg \log x$$

(means, $\exists\, c > 0$ s.t. $\pi(x) \geq C \log x$ for all large x).

<u>This course:</u> $\quad \pi(x) \sim x / \log x$

7

$$\pi(x) = \int_2^x \frac{dt}{\log t} + O\left(x \exp(-c\sqrt{\log x})\right).$$

Next: Riemann - Stieltjes integration

& properties of Dirichlet series.

§2. Analytic properties of Dirichlet series.

Recall — looking at Dirichlet series of shape

$$\alpha(s) = \sum_{n=1}^{\infty} a_n n^{-s} \qquad (a_n \in \mathbb{C}). \quad \text{——} \quad (2.1)$$

Today: analogue of radius of convergence for power series, basic properties. Will use Riemann-Stieltjes integration.

Definition 2.1 (i) The abscissa of convergence of a Dirichlet series $\alpha(s)$ as in (2.1) is

$$\sigma_c := \inf \left\{ \sigma \in \mathbb{R} : \sum_{n=1}^{\infty} a_n n^{-s} \text{ converges for all } s \text{ with } \mathrm{Re}(s) > \sigma \right\};$$

(ii) the line $\sigma = \sigma_c$ is ~~the~~ the line of convergence of $\alpha(s)$;

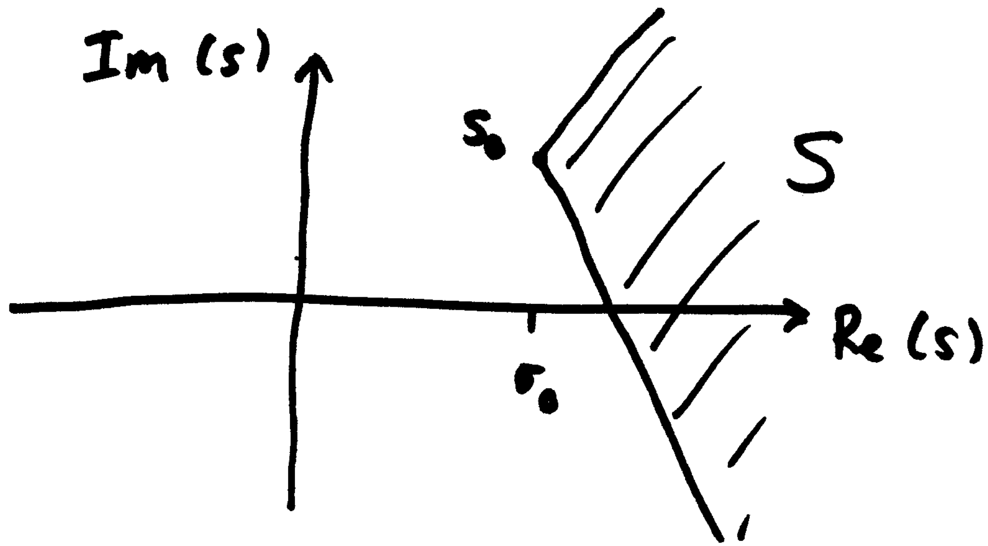
(iii) the half-plane of convergence of $\alpha(s)$ is

$$H_c = \{ s \in \mathbb{C} : \mathrm{Re}(s) > \sigma_c \}.$$

②

<u>**Theorem 2.2.**</u> Suppose that $\alpha(s) = \sum_{n=1}^{\infty} a_n n^{-s}$ converges at the point $s = s_0$. Then whenever $H > 0$, the series $\alpha(s)$ is uniformly convergent in the sector

$$S = \{ s \in \mathbb{C} : \sigma \geq \sigma_0, \ |t - t_0| \leq H(\sigma - \sigma_0) \}$$



[Note: take $H \longrightarrow \infty$ to get a half-plane
$\mathrm{Re}(s) > \sigma_0$]

③

<u>Corollary 2.3.</u> The abscissa of convergence of $\alpha(s)$ has the property that $\alpha(s)$ converges for all $s \in \mathbb{C}$ with $\sigma > \sigma_c$, and for no $s \in \mathbb{C}$ with $\sigma < \sigma_c$. Further, when $s_0 \in \mathbb{C}$ satisfies with $\sigma_0 > \sigma_c$, then there is a neighbourhood of s_0 in which $\alpha(s)$ converges uniformly.

Proof involves Riemann - Stieltjes integration.

Define $\int_a^b f(x)\, d\, g(x)$ as a limit of Riemann sums $\sum_n f(\xi_n) \Delta g(x_n)$:

When $a < b$ and $a = x_0 \leq x_1 \leq \ldots \leq x_N = b$ and for each n we have ξ_n with $x_{n-1} \leq \xi_n \leq x_n$, 4 We put

$$S(\underline{x}, \underline{\xi}) = \sum_{n=1}^{N} f(\xi_n)\left(g(x_n) - g(x_{n-1})\right).$$

We say that the Riemann–Stieltjes integral $\int_a^b f(x)\,dg(x)$ exists with value I if, for each $\varepsilon > 0$, there exists $\delta > 0$ such that

$$\left| S(\underline{x}, \underline{\xi}) - I \right| < \varepsilon,$$

whenever

$$\text{mesh}\,\{\underline{x}\} := \max_{1 \le n \le N} (x_n - x_{n-1}) \le \delta.$$

One can check as exercises the following:

A The R–S integral $\int_a^b f(x)\,dg(x)$ exists whenever f is continuous on $[a,b]$ and

⑤

g is bounded variation on $[a,b]$.

$$\left[\text{Thus } \mathrm{Var}_{[a,b]}(g) = \sup_{\underline{x}} \sum_{n=1}^{N} |g(x_n) - g(x_{n-1})| < \infty \right]$$

(B) If $\int_a^b f\,dg$ exists, then so too does $\int_a^b g\,df$,

and $$\int_a^b g\,df = f(b)g(b) - f(a)g(a) - \int_a^b f\,dg.$$

(C) If g' is cts on $[a,b]$, then

$$\int_a^b f\,dg = \int_a^b f g'\,dx.$$

* (D) Suppose that f is continuous on $[0,M]$ and $(a_n)_{n=1}^{\infty}$ is a sequence of complex numbers. Then on writing $A(x) = \sum_{1 \le m \le x} a_m$,

we have

$$\sum_{m=1}^{M} a_m f(m) = \int_0^M f(x)\, d\, A(x).$$

Moreover, if f' is continuous on $[0, M]$, then

$$\sum_{m=1}^{M} a_m f(m) = A(M) f(M) - \int_0^M A(x) f'(x)\, dx.$$

[cf. Partial summation] $\longleftrightarrow$ Abel summation.

Sketch of (D): if mesh $\{\underline{x}\} \leq \delta$ then:

$A(x_n) - A(x_{n-1}) = 0$ whenever x_n, x_{n-1} both lie within an interval between successive integers.

and

$A(x_n) - A(x_{n-1}) = a_m$ when $x_{n-1} < m < x_n$, with $m \in \mathbb{Z} \cap [0, M]$.

Thus, cts property of f implies ~~that~~ we may suppose $|f(\xi_n) - f(m)| < \varepsilon^2$ when $x_{n-1} < m < x_n$, so

$$\left| S(\underline{x}, \underline{\xi}) - \sum_{m=1}^{M} a_m f(m) \right| < \varepsilon^2 \sum_{m=1}^{M} |a_m| < \varepsilon,$$

say.

__Proof of Theorem 2.2.__ We consider the tail of infinite series defining $\alpha(s)$. Put

$$R(u) = \sum_{n>u} a_n n^{-s_0}.$$

Use R-S to move from s_0 to s. We have

$$\sum_{n=M+1}^{N} a_n n^{-s} \overset{(A),(D)}{=} -\int_M^N u^{s_0-s} \, dR(u)$$

⑧

$$\overset{(B)}{=} \; - u^{s_0 - s} R(u) \Big|_M^N + \int_M^N R(u) \, d\, u^{s_0 - s}$$

$$\overset{(c)}{=} \; M^{s_0 - s} R(M) - N^{s_0 - s} R(N)$$

$$+ (s_0 - s) \int_M^N R(u) \, u^{s_0 - s - 1} \, du$$

(2.2)

Since $\alpha(s)$ converges at $s = s_0$, we may suppose that $M = M(\varepsilon)$ is chosen so that $|R(u)| \leq \varepsilon$ for all $u \geq M$. Then, whenever $\sigma > \sigma_0$, we have

$$\Big| \sum_{n=M+1}^{N} a_n n^{-s} \Big| \leq 2\varepsilon M^{\sigma_0 - \sigma} + |s_0 - s| \int_M^\infty |R(u)| \, u^{\sigma_0 - \sigma -} du$$

$$\leq 2\varepsilon M^{\sigma_0 - \sigma} + \varepsilon |s_0 - s| \int_M^\infty u^{\sigma_0 - \sigma - 1} du$$

⑨

$$\leqslant 2\varepsilon + \varepsilon \frac{|s_0 - s|}{\sigma - \sigma_0}$$

When $|t - t_0| \leq H(\sigma - \sigma_0)$, we have

$$|s - s_0| \leq \sigma - \sigma_0 + |t - t_0| \leq (H+1)(\sigma - \sigma_0).$$

Thus

$$\left| \sum_{n=M+1}^{N} a_n n^{-s} \right| \leqslant (H+3)\varepsilon$$

(so "tail" of $\alpha(s)$ is small)

Thus (take $N \to \infty$) we conclude ~~that~~ $\alpha(s)$ converges uniformly in $\mathbb{S}$. //

Recall: analytic properties of Dirichlet series

<u>Corollary</u> $\underline{2\cdot 3}$ The abscissa of convergence of a Dirichlet series $\alpha(s) = \sum_{n=1}^{\infty} a_n n^{-s}$ has the property that

- $\alpha(s)$ converges for all $s \in \mathbb{C}$ with $\sigma > \sigma_c$,
- $\alpha(s)$ converges for no $s \in \mathbb{C}$ with $\sigma < \sigma_c$.

Further, when $s_0 \in \mathbb{C}$ satisfies $\sigma_0 > \sigma_c$, then there is a nhd of s_0 in which $\alpha(s)$ converges uniformly.

Proof (via Theorem 2.2) uses Riemann-Stieltjes integration. In particular, with $R(u) = \sum_{n>u} a_n n^{-s_0}$, have

$$\sum_{n=M+1}^{N} a_n n^{-s} = M^{s_0-s} R(M) - N^{s_0-s} R(N) + (s_0-s)\int_M^N R(u) u^{s_0-s-1} du \qquad (2\cdot 2)$$

---

n^{-s} is analytic function of s $\quad (n^{-s} = e^{-s\log n})$

and $\alpha(s) = \sum_{n=1}^{\infty} a_n n^{-s}$ is locally uniformly convergent for $\sigma > \sigma_c$ $\Rightarrow$

$\alpha(s)$ analytic for $\sigma > \sigma_c$.

Similarly, $\alpha'(s) = -\sum_{n=1}^{\infty} a_n (\log n) n^{-s}$

is locally uniformly convergent for $\sigma > \sigma_c$ $\Rightarrow$ analytic for $\sigma > \sigma_c$.

<u>Question:</u> How does one compute σ_c from $(a_n)_{n=1}^{\infty}$?

[Analogous to computing radius of convergence of a power series].

3

Theorem 2.4. Let $\alpha(s) = \sum_{n=1}^{\infty} a_n n^{-s}$ and write

$$A(x) = \sum_{n \le x} a_n .$$

(i) When $\sigma_c \ge 0$, one has $\sigma_c = \limsup_{x \to \infty} \frac{\log |A(x)|}{\log x}$;

(ii) When $\sigma_c < 0$, the function $A(x)$ is bounded;

(iii) When $\sigma > \max\{\sigma_c, 0\}$, one has

$$\sum_{n=1}^{\infty} a_n n^{-s} = s \int_1^{\infty} \frac{A(x)}{x^{s+1}}\, dx . \qquad (2.3)$$

Proof: Put

$$\sigma^* := \limsup_{x \to \infty} \frac{\log |A(x)|}{\log x} .$$

If $\sigma_0 > \sigma^*$, then $A(x) \ll x^{\sigma_0}$ (with an implicit constant depending on $\underline{a}$ and σ_0).

④ Then whenever $\sigma > \sigma_0$, we see that

$$\int_1^\infty \left|\frac{A(x)}{x^{s+1}}\right| dx \ll \int_1^\infty \frac{x^{\sigma_0}}{x^{\sigma+1}}\, dx < \infty.$$

Meanwhile, by Riemann - Stieltjes integration,

$$\sum_{n=1}^N a_n n^{-s} = \int_{1^-}^N x^{-s}\, dA(x)$$

$$= A(x)x^{-s}\Big|_{1^-}^N - \int_{1^-}^N A(x)\, d\, x^{-s}$$

$$= \underbrace{A(N)\,N^{-s}}_{\ll N^{\sigma_0-\sigma} \to 0 \text{ as } N\to\infty} + s\int_1^N A(x)\,x^{-s-1}\,dx$$

↑ absolutely convergent.

Then $\sum_{n=1}^\infty a_n n^{-s} = s\int_1^\infty A(x)x^{-s-1}dx.$ ——— (2.4)

⑤

holds for $\sigma > \sigma^*$.

Divide into cases:

(a) Suppose that $\sigma_c < 0$: Cor. 2·3 (with $s = 0$) shows that $A(x)$ converges as $x \to \infty$, whence $\sigma^* \leq 0$. Thus $A(x)$ is bounded and (2·3) holds for $\sigma > 0$. This proves (ii). □

(b) Suppose $\sigma_c \geq 0$. Then Cor. 2·3 shows that $\alpha(s)$ diverges when $\sigma < \sigma_c$, whence $\sigma^* \geq \sigma_c$. To show that $\sigma^* \leq \sigma_c$, consider $\sigma_0 > \sigma_c$ and note (2·2) gives

$$A(N) = -R(N)N^{\sigma_0} + \sigma_0 \int_0^N R(u)\, u^{\sigma_0 - 1}\, du \quad (M=0 \,\&\, s=0)$$

where $R(u) = \sum_{n>u} a_n n^{-s_0}$.

⑥ This series is bounded because $\alpha(s_0)$ converges and hence $A(N) \ll_{\underline{a}, \sigma_0} N^{\sigma_0}$. But then $\sigma^* \leq \sigma_0$.

This holds whenever $\sigma_0 > \sigma_c$, so $\sigma^* \leq \sigma_c$. Then $\sigma^* = \sigma_c$, proves (i), and also (iii) in view of (2.4). //

<u>Definition 2.5.</u> The <u>abscissa</u> of <u>absolute convergence</u> of a Dirichlet series $\alpha(s) = \sum_{n=1}^{\infty} a_n n^{-s}$ is

$$\sigma_a := \inf \left\{ \sigma \in \mathbb{R} : \sum_{n=1}^{\infty} \frac{|a_n|}{n^{\sigma}} < \infty \right\}$$

[This is abscissa of convergence of $\sum_{n=1}^{\infty} \frac{|a_n|}{n^{+s}}$]

Theorem 2.6. For any Dirichlet series $\alpha(s) = \sum_{n=1}^{\infty} a_n n^{-s}$, one has $\sigma_c \le \sigma_a \le \sigma_c + 1$.

Proof: Problem Set 1.

Theorem 2.7. Suppose $\alpha(s) = \sum_{n=1}^{\infty} a_n n^{-s}$ has abscissa of convergence σ_c. Suppose that $\delta > 0$ and $\sigma \geq \sigma_c + \delta$. Then for each $\varepsilon > 0$, with $\varepsilon < \delta$, one has

$$\alpha(s) \ll_{\underline{a}, \delta, \varepsilon} (|t| + 4)^{1 - \delta + \varepsilon}.$$

Proof: see notes.

Theorem 2.8. Suppose that for all s with $\sigma > \sigma_0$, one has

$$\sum_{n=1}^{\infty} a_n n^{-s} = \sum_{n=1}^{\infty} b_n n^{-s}.$$

⑧ Then $a_n = b_n$ for all $n \in \mathbb{N}$.

<u>Proof.</u> Put $c_n = a_n - b_n$ and consider

$$\sum_{n=1}^{\infty} c_n n^{-s}.$$

We prove by induction ~~that~~ $c_n = 0$ for all n.

We may suppose ~~that~~ $\sigma > \sigma_0$ and $\sum_{n=1}^{\infty} c_n n^{-\sigma} = 0$.

Thus

$$c_1 = -\sum_{n=2}^{\infty} c_n n^{-\sigma} \longrightarrow 0 \text{ as } \sigma \to \infty$$

abs. conv. for $\sigma > \sigma_0 + 1$

Then $|c_1| \leq \sum_{n=2}^{\infty} |c_n| n^{-\sigma} \longrightarrow 0$ as $\sigma \to \infty$

$\Downarrow$

$c_1 = 0$. (Base of induct^{n}.)

Next, if $c_n = 0$ for $n < N$ and $N \geq 2$,

(9) then

$$|c_N| \leq \sum_{n>N} |c_n| (N/n)^\sigma \longrightarrow 0 \quad \text{as } \sigma \to \infty.$$

$\downarrow$

$c_N = 0$. Result follows by ind^n. //

31 Aug 2020

①

Recall: properties of Dirichlet series $\alpha(s) = \sum_{n=1}^{\infty} a_n n^{-s}$

$\sigma_c = \inf \{ \sigma \in \mathbb{R} : \alpha(s) \text{ converges for all } \mathrm{Re}(s) > \sigma \}$

$\sigma_a = \inf \{ \sigma \in \mathbb{R} : \sum_{n=1}^{\infty} |a_n| n^{-\sigma} < \infty \}$

Thm 2.6: $\sigma_c \leq \sigma_a \leq \sigma_c + 1$

Thm 2.5 if $\sigma_c \geq 0$, then $\sigma_c = \limsup_{x \to \infty} \dfrac{\log |A(x)|}{\log x}$, $A(x) = \sum_{n \leq x} a_n$

Thm 2.8 Suppose for all large σ have $\sum_{n=1}^{\infty} a_n n^{-s} = \sum_{n=1}^{\infty} b_n n^{-s}$.
Then $a_n = b_n$ all n.

§3. The Riemann zeta function: basic properties.

Define $\zeta(s) = \sum_{n=1}^{\infty} n^{-s}$ for $\mathrm{Re}(s) > 1$ $(\sigma_c = 1)$.

First justify Euler product for $\zeta(s)$ and other cousins.
Begin by justifying multiplicative property of Dirichlet series.

②

Theorem 3.1. Let $\alpha(s) = \sum_{n=1}^{\infty} a_n n^{-s}$ and $\beta(s) = \sum_{n=1}^{\infty} b_n n^{-s}$.

Put $\gamma(s) = \sum_{n=1}^{\infty} c_n n^{-s}$, where $c_n = \sum_{lm=n} a_l b_m$.

Then provided that $\alpha(s)$ and $\beta(s)$ are both absolutely convergent at $s \in \mathbb{C}$, so too is $\gamma(s)$, and $\gamma(s) = \alpha(s)\beta(s)$.

Proof. Absolute convergence of $\alpha(s)$ and $\beta(s)$ implies can rearrange terms to get

$$\alpha(s)\beta(s) = \sum_{l=1}^{\infty}\sum_{m=1}^{\infty} a_l b_m (lm)^{-s} = \sum_{n=1}^{\infty} c_n n^{-s} = \gamma(s),$$

with $\gamma(s)$ abs. conv:

$$\sum_{n=1}^{\infty} |c_n n^{-s}| \le \sum_{n=1}^{\infty} \sum_{lm=n} |a_l b_m| (lm)^{-\sigma}$$

$$= \left(\sum_{l=1}^{\infty} |a_l| l^{-\sigma}\right)\left(\sum_{m=1}^{\infty} |b_m| m^{-\sigma}\right)$$

$$< \infty. \quad //$$

③ "Recall": arithmetic function $f : \mathbb{N} \longrightarrow \mathbb{C}$ is multiplicative if $f(1) = 1$, and, whenever $(m,n)=1$ then $f(mn) = f(m) f(n)$.

[is totally multiplicative if $f(mn) = f(m) f(n)$ for all $m, n \in \mathbb{N}$]

Important property: If $n = \prod_{p^h \| n} p^h$ is the standard factorisation of $n \in \mathbb{N}$ into distinct primes p, where notation: $p^h \| n$ means $p^h \mid n$ & $p^{h+1} \nmid n$, then

$$f \text{ multiplicative} \Rightarrow \boxed{f(n) = \prod_{p^h \| n} f(p^h).} \; *$$

$$\left[f \text{ totally multiplicative} \Rightarrow f(n) = \prod_{p^h \| n} f(p)^h \right]$$

④

Examples :

(i) $f(n) = \mathbb{1}(n) = \{1, \quad n \in \mathbb{N}$

(ii) (Möbius function) $\mu(n) = \begin{cases} (-1)^{\omega(n)}, & n \text{ squarefree} \\ 0 & \text{otherwise}, \end{cases}$

where $\omega(n) = \sum_{p \mid n} 1$.

(iii) (Liouville function) $\lambda(n) = \{(-1)^{\Omega(n)} \quad (n \in \mathbb{N})$

where $\Omega(n) = \sum_{p^h \| n} h$

(iv) (Divisor function) $\tau(n) = \sum_{\substack{d \mid n \\ (d \in \mathbb{N})}} 1$.

Check : all are multiplicative.

5

Theorem 3.2. Let f be multiplicative and write
$$\varphi(s) = \sum_{n=1}^{\infty} f(n) n^{-s}.$$
Then whenever $\varphi(s)$ is absolutely convergent, one has the <u>Euler product</u> decomposition
$$\varphi(s) = \prod_p \sum_{h=0}^{\infty} \frac{f(p^h)}{p^{hs}}$$
$$= \prod_p \left(1 + \frac{f(p)}{p^s} + \frac{f(p^2)}{p^{2s}} + \cdots \right)$$

<u>Proof:</u> Suppose that $\varphi(s)$ is absolutely convergent at $s \in \mathbb{C}$. Each factor in the Euler product is absolutely convergent, since
$$\sum_{h=0}^{\infty} \left| \frac{f(p^h)}{p^{hs}} \right| \leq \sum_{n=1}^{\infty} \frac{|f(n)|}{n^{\sigma}} < \infty.$$

We may therefore rearrange terms in any finite product of such Euler factors. Put

$$\mathcal{N}(x) = \{ n \in \mathbb{N} : p \mid n \Rightarrow p \le x \}.$$

Then

$$\prod_{p \le x} \sum_{h=0}^{\infty} \frac{f(p^h)}{p^{hs}} = \sum_{n \in \mathcal{N}(x)} \frac{f(n)}{n^s},$$

$|f(n)/n^s|$

Whence

$$\left| \sum_{n=1}^{\infty} \frac{f(n)}{n^s} - \prod_{p \le x} \sum_{h=0}^{\infty} \frac{f(p^h)}{p^{hs}} \right| \le \sum_{n \notin \mathcal{N}(x)} \frac{|f(n)|}{n^\sigma}$$

$$\le \sum_{n > x} \frac{|f(n)|}{n^\sigma}.$$

But given any $\varepsilon > 0$, we can take x large enough so ~~that~~ rhs is smaller ε (since $\varphi(s)$ is abs. conv).

Thus,

⑦ $$\lim_{x\to\infty} \prod_{p\le x} \sum_{h=0}^{\infty} \frac{f(p^h)}{p^{hs}} = \sum_{n=1}^{\infty} \frac{f(n)}{n^s} = \varphi(s).$$ //

Corollary 3.3: When $\sigma > 1$, one has

(i) $$\sum_{n=1}^{\infty} n^{-s} = \zeta(s) = \prod_p (1-p^{-s})^{-1} \ ; \ ✓$$

(ii) $$\sum_{n=1}^{\infty} \mu(n) n^{-s} = 1/\zeta(s) = \prod_p (1-p^{-s}) \ ;$$

(iii) $$\sum_{n=1}^{\infty} \lambda(n) n^{-s} = \frac{\zeta(2s)}{\zeta(s)} = \prod_p (1+p^{-s}) \ ;$$

(iv) $$\sum_{n=1}^{\infty} \tau(n) n^{-s} = \zeta(s)^2 = \prod_p (1-p^{-s})^{-2}.$$

Proof: When $\sigma > 1$, the series in (i), (ii), (iii) are abs. conv., since $\sum n^{-\sigma} < \infty$. The

(8) series in (iv) is therefore abs. conv. by virtue of Theorem 3.1.:

$$\zeta(s)^2 = \sum_{n=1}^{\infty} \Big(\underbrace{\sum_{lm=n} 1}_{= \tau(n)} \Big) n^{-s}.$$

The coeffts of (i) - (iv) are all multiplicative, so in cases (i) & (ii) we obtain Euler product:

$$\zeta(s) = \prod_p \sum_{h=0}^{\infty} \frac{\mathbb{1}(p^h)}{p^{hs}} = \prod_p \left(1 + p^{-s} + p^{-2s} + \dots\right)$$

$$= \prod_p \left(1 - p^{-s}\right)^{-1}, \quad \square$$

$$\sum_{n=1}^{\infty} \frac{\mu(n)}{n^s} = \prod_p \sum_{h=0}^{\infty} \frac{\mu(p^h)}{p^{hs}} = \prod_p \left(1 - \frac{1}{p^s}\right).$$

9

Thus

$$\zeta(s)\sum_{n=1}^{\infty}\frac{\mu(n)}{n^s} = \prod_p\left(1-p^{-s}\right)^{-1}\cdot\prod_p\left(1-p^{-s}\right)$$
$$= 1.$$

$$\Rightarrow \sum_{n=1}^{\infty}\frac{\mu(n)}{n^s} = \frac{1}{\zeta(s)}. \quad \square$$

In case (iii) we obtain similarly

$$\sum_{n=1}^{\infty}\frac{\lambda(n)}{n^s} = \prod_p\left(1-p^{-s}+p^{-2s}-\cdots\right) = \prod_p\left(1+p^{-s}\right)^{-1}$$

Thus

$$\zeta(s)\sum_{n=1}^{\infty}\frac{\lambda(n)}{n^s} = \prod_p\left(1-p^{-s}\right)^{-1}\left(1+p^{-s}\right)^{-1} = \prod_p\left(1-p^{-2s}\right)^{-1}$$
$$= \zeta(2s)$$

$$\Rightarrow \sum_{n=1}^{\infty}\frac{\lambda(n)}{n^s} = \frac{\zeta(2s)}{\zeta(s)}. \quad \square$$

Finally,

⑩

$$\sum_{n=1}^{\infty} \frac{\tau(n)}{n^s} = \prod_p \sum_{h=0}^{\infty} \frac{\tau(p^h)}{p^{hs}} = \prod_p \left(1 + 2p^{-s} + 3p^{-2s} + \cdots\right)$$

$$= \prod_p (1-p^{-s})^{-2} = \zeta(s)^2 . \square \quad //$$

2 SEP 2020

①

HOMEWORK SUBMISSION — go to Brightspace course website.

Go to COURSE TOOLS
|
ASSIGNMENTS
|
Select PROBLEM SHEET 1

} upload pdf scan of homework solutions

Recall: absolute convergence and multiplication of Dirichlet series,
Euler products $\longleftrightarrow$ multiplicative functions

$$\zeta(s) = \sum_{n=1}^{\infty} n^{-s} = \prod_p (1-p^{-s})^{-1} \qquad (\sigma > 1)$$

$$\frac{1}{\zeta(s)} = \sum_{n=1}^{\infty} \mu(n) n^{-s} = \prod_p (1-p^{-s}) \qquad \text{"}$$

$$\frac{\zeta(2s)}{\zeta(s)} = \sum_{n=1}^{\infty} \lambda(n) n^{-s} = \prod_p (1+p^{-s})^{-1} \qquad \text{"}$$

$$\zeta(s)^2 = \sum_{n=1}^{\infty} \tau(n) n^{-s} = \prod_p (1-p^{-s})^{-2}$$

② Connection with Möbius inversion formula.

Suppose f, g arithmetic functions. Then

$$g(n) = \sum_{d|n} f(d) \iff f(n) = \sum_{d|n} \mu(d) g(n/d)$$

$$\sum_{dk=n} \mathbb{1}(d) f(k)$$

Compare:

$$\sum_{n=1}^{\infty} g(n) n^{-s} = \sum_{n=1}^{\infty} \left(\sum_{d|n} f(d) \right) n^{-s} = \zeta(s) \sum_{n=1}^{\infty} f(n) n^{-s}$$

$$\Updownarrow$$

$$\sum_{n=1}^{\infty} f(n) n^{-s} = \zeta(s)^{-1} \sum_{n=1}^{\infty} g(n) n^{-s} = \sum_{n=1}^{\infty} \left(\sum_{d|n} \mu(d) g(n/d) \right) n^{-s}.$$

Dirichlet convolution of a & b: $\quad a * b(n) = \sum_{d|n} a(d) b(n/d)$

Define the **von Mangoldt function** $\Lambda(n)$:

$$\Lambda(n) = \begin{cases} \log p, & \text{when } n = p^h, \ h \geq 1, \ p \text{ prime} \\ 0, & \text{otherwise}. \end{cases}$$

③

Corollary 3.4. When $\sigma > 1$, one has

$$\log \zeta(s) = \sum_{n=2}^{\infty} \frac{\Lambda(n)}{\log n} n^{-s}$$

and

$$-\frac{\zeta'(s)}{\zeta(s)} = \sum_{n=1}^{\infty} \Lambda(n) n^{-s}.$$

Proof. Use Euler product for $\zeta(s)$:

$$\log \zeta(s) = -\sum_{p} \log(1-p^{-s}) = \sum_{p} \sum_{k=1}^{\infty} k^{-1} p^{-ks}$$

$$= \sum_{n=2}^{\infty} \frac{\Lambda(n)}{\log n} n^{-s} \qquad (\text{for } \sigma > 1).$$

By differentiation:

$$\frac{\zeta'(s)}{\zeta(s)} = -\sum_{n=2}^{\infty} \frac{\Lambda(n)}{\log n} \log n . n^{-s}$$

$$= -\sum_{n=2}^{\infty} \Lambda(n) n^{-s}. \quad //$$

(4) <u>Analytic continuation of Riemann zeta function.</u>

So far, we've defined $\zeta(s)$ to be $\sum_{n=1}^{\infty} n^{-s}$ when $\sigma > 1$. Natural to attempt to define $\zeta(s)$ for $\sigma \leq 1$.

<u>Theorem 3.5.</u> Suppose ~~that~~ $s \in \mathbb{C}$ satisfies $s \neq 1$, $\sigma > 0$, and that $x > 0$. Then one has

$$\zeta(s) = \sum_{n \leq x} n^{-s} + \frac{x^{1-s}}{s-1} + \frac{\{x\}}{x^s} - s\int_x^{\infty} \frac{\{u\}}{u^{s+1}}\, du.$$

[<u>Note</u>: $\{\theta\} = \theta - \lfloor \theta \rfloor$, so integral on rhs is abs. conv.]

<u>Proof</u>: Noting that for $\sigma > 1$, one has

$$\zeta(s) = \sum_{n=1}^{\infty} n^{-s} = \sum_{1 \leq n \leq x} n^{-s} + \sum_{n > x} n^{-s}, \qquad (3.1)$$

we may apply R-S integration on final sum to get

$$\sum_{n>x} n^{-s} = \int_{x^+}^{\infty} u^{-s}\, d\lfloor u \rfloor = \int_{x^+}^{\infty} u^{-s}\, du - \int_{x^+}^{\infty} u^{-s}\, d\{u\}$$

$$(\lfloor u \rfloor = u - \{u\}.)$$

$$= \frac{x^{1-s}}{s-1} + \{x\} x^{-s} + \int_{x}^{\infty} \{u\}\, du^{-s}. \qquad (3.2)$$

The formula claimed in theorem follows for $\sigma > 1$ by combining (3.1) and (3.2).

Since the integral $\int_x^\infty \frac{\{u\}}{u^{s+1}}\, du$ is abs. conv. for $\sigma > 0$, and uniformly for $\sigma > \delta > 0$, and the integrand is an analytic function of s, it follows that integral is analytic for $\sigma > 0$. Thus

⑥ by uniqueness of analytic continuations, the claimed formula for $\zeta(s)$ holds for $\sigma > 0$. //

(Can continue by integration by parts / Euler-MacLaurin summation formula).

Corollary 3.6 The Riemann zeta function has a simple pole at $s=1$ with residue 1, but is otherwise analytic for $\sigma > 0$.

Proof. Apply Theorem 3.5 with $x=1$:

$$\zeta(s) = 1 + \frac{1}{s-1} - s\int_1^\infty \frac{\{u\}}{u^{s+1}}\,du$$

$$= \frac{s}{s-1} - s\int_1^\infty \frac{\{u\}}{u^{s+1}}\,du \quad (\sigma > 0). \tag{3.7}$$

⑦

Thus $\zeta(s)$ has simple pole at $s=1$, residue 1 at $s=1$. //

4 Sep 2020

① **Recall:** $$-\frac{\zeta'(s)}{\zeta(s)} = \sum_{n=1}^{\infty} \frac{\Lambda(n)}{n^s}, \quad \Lambda(n) = \begin{cases} \log p, & n = p^h \\ 0, & \text{else} \end{cases}$$

Thus $$-\zeta'(s) = \zeta(s)\left(-\frac{\zeta'}{\zeta}(s)\right) = \sum_{n=1}^{\infty}\left(\sum_{d|n} \Lambda(n/d)\right) n^{-s}$$

$$-\zeta'(s) = \sum_{n=1}^{\infty} (\log n) n^{-s}$$

$$\Rightarrow \quad \log n = \sum_{d|n} \Lambda(n/d) = \sum_{d|n} \Lambda(d)$$

$d \longleftrightarrow n/d$

Theorem 3.5 Suppose $s \in \mathbb{C}\setminus\{1\}$ satisfies $\sigma > 0$. Then whenever $x > 0$, one has

$$\zeta(s) = \sum_{n \le x} n^{-s} + \frac{x^{1-s}}{s-1} + \frac{\{x\}}{x^s} - s\int_x^{\infty} \frac{\{u\}}{u^{s+1}}\, du.$$

Corollary 3.6: The Riemann zeta function has a simple pole at $s=1$ with residue 1, but is otherwise analytic for $\sigma > 0$.

Proof: Thm 3.5 $\Rightarrow$ $$\zeta(s) = \frac{s}{s-1} - s\int_1^{\infty} \frac{\{u\}}{u^{s+1}}\, du \quad (\sigma > 0). \qquad \text{— (3.3)}$$

② **Corollary 3.7** Let C_0 denote Euler's constant (often called $\gamma = 0.577\ldots$), defined by

$$C_0 = \lim_{x\to\infty} \Big(\sum_{1\le n\le x} \frac{1}{n} - \log x \Big).$$

Then the Laurent expansion of $\zeta(s)$ about $s=1$ is given by

$$\zeta(s) = \frac{1}{s-1} + \sum_{k=0}^{\infty} a_k (s-1)^k,$$

in which $a_0 = C_0$.

Proof. Euler's constant: apply R-S integration to get

$$\sum_{1\le n\le x} \frac{1}{n} = \int_{1^-}^{x} u^{-1} d\Big(\underbrace{\sum_{n\le u} 1}_{\lfloor u\rfloor = u - \{u\}} \Big) = \int_{1^-}^{x} u^{-1}\,du - \int_{1^-}^{x} u^{-1}\, d\{u\}$$

$$= \log x + 1 - \{x\}/x - \int_{1^-}^{x} u^{-2}\{u\}\,du$$

③ So
$$\int_x^\infty u^{-2}\{u\}\,du = O\left(\frac{1}{x}\right)$$
$$\sum_{1\le n\le x}\frac{1}{n} = \log x + c_0 + O\left(\frac{1}{x}\right), \qquad (3.4)$$

where
$$c_0 = 1 - \int_1^\infty \{u\}\,u^{-2}\,du.$$

Compare (3.4) & defn of C_0, we ~~see that~~ $c_0 = C_0$.
Also, by comparison with (3.3), we ~~see that~~
$$\zeta(s) = \frac{1}{s-1} + \left(1 - \int_1^\infty \frac{\{u\}}{u^2}\,du\right) + O(|s-1|)$$

Thus
$$\zeta(s) = \frac{1}{s-1} + C_0 + O(|s-1|),$$ confirms Laurent series expansion. //

④

§ 4. Arithmetic functions — mean values and elementary estimates.

$$f : \mathbb{N} \longrightarrow \mathbb{C}$$

$$\sum_{n=1}^{\infty} f(n) n^{-s} \qquad \boxed{\frac{1}{x} \sum_{1 \le n \le x} f(n)} \longleftrightarrow \sigma_c$$

Standard strategy: Suppose $g : \mathbb{N} \to \mathbb{C}$

$$f(n) = \sum_{d \mid n} g(d), \tag{4·1}$$

so that

$$\left.\begin{aligned} \frac{1}{x} \sum_{1 \le n \le x} f(n) &= \frac{1}{x} \sum_{1 \le n \le x} \sum_{d \mid n} g(d) \\ &= \frac{1}{x} \sum_{1 \le d \le x} g(d) \sum_{\substack{1 \le n \le x \\ d \mid n}} 1 \end{aligned}\right\} \circledast$$

$$\sum_{\substack{1 \le n \le x \\ d \mid n}} \;\updownarrow\; 1 \le m \le x/d \quad (n = md)$$

Then

$$\frac{1}{x}\sum_{1\le n\le x} f(n) = \frac{1}{x}\sum_{1\le d\le x} g(d)\left\lfloor \frac{x}{d}\right\rfloor$$

$$= \frac{1}{x}\sum_{1\le d\le x}\left(\frac{x}{d}+O(1)\right)g(d)$$

$$= \sum_{1\le d\le x}\frac{g(d)}{d} + O\left(\frac{1}{x}\sum_{1\le d\le x}|g(d)|\right)$$

often this is better behaved than $\sum f(n)$

often well-controlled.

<u>Möbius inversion</u>: Given $f:\mathbb{N}\to\mathbb{C}$, we can obtain a function $g:\mathbb{N}\to\mathbb{C}$ ~~satisfying~~ (4.1) via

$$f(n) = \sum_{d|n} g(d) \iff g(n) = \sum_{d|n}\mu(d)f(n/d)$$

<u>Example 4.1</u> Let $\sigma(n) := \sum_{d\mid n} d$. Then one has

$$\sum_{n\le x}\frac{\sigma(n)}{n} = \sum_{n\le x}\frac{1}{n}\sum_{d\mid n} d = \sum_{n\le x}\sum_{d\mid n}\frac{1}{d} \qquad \frac{n}{d}\longleftrightarrow d$$

$$= \sum_{1\le d\le x}\frac{1}{d}\sum_{1\le m\le x/d} 1 = \sum_{1\le d\le x}\frac{1}{d}\left\lfloor\frac{x}{d}\right\rfloor$$

$$= \sum_{1\le d\le x}\frac{x}{d^2} + O\Big(\sum_{1\le d\le x}\frac{1}{d}\Big)$$

$$= x\Big(\sum_{d=1}^{\infty}\frac{1}{d^2} + O\Big(\underbrace{\sum_{d>x}\frac{1}{d^2}}_{=O(1/x)}\Big)\Big) + O(\log x)$$

$$= x\,\zeta(2) + O(\log x)$$

Then $$\frac{1}{x}\sum_{n\le x}\frac{\sigma(n)}{n} = \underbrace{\frac{\pi^2}{6}}_{=\zeta(2)} + O\Big(\frac{\log x}{x}\Big). \qquad \square$$

<u>Example 4.2.</u> Compute the average value of $n/\varphi(n)$, when $\varphi(n) = n\prod_{p|n}(1-1/p)$.

We seek $g : \mathbb{N} \to \mathbb{C}$ with $\sum_{d|n} g(d) = n/\varphi(n)$

By Möbius inversion $$g(n) = \sum_{d|n} \mu(d)\,\frac{n/d}{\varphi(n/d)}.$$

But μ, φ, n are all mult. fns of n, so too is $g(n)$ [if a, b are mult, so too is $\sum_{d|n} a(d)b(n/d)$].

So suffices to evaluate g at prime powers:

$$g(1) = 1$$

$$g(p) = \sum_{d|p} \mu(d)\frac{p/d}{\varphi(p/d)} = \frac{p}{p-1} - 1 = \frac{1}{p-1} = \frac{1}{\varphi(p)}$$

⑧ When $h \geq 2$,

$$g(p^h) = \sum_{l=0}^{h} \mu(p^l) \frac{p^h/p^l}{\varphi(p^h/p^l)} = \frac{p^h}{p^h(1-1/p)} - \frac{p^{h-1}}{p^{h-1}(1-\frac{1}{p})}$$

$$= 0 \quad (\text{all for } h \geq 2)$$

Thus

$$g(n) = \prod_{p^h \| n} g(p^h) = \frac{\mu(n)^2}{\varphi(n)}$$

Next, we deduce

$$\sum_{1 \leq n \leq x} \frac{n}{\varphi(n)} = \sum_{1 \leq n \leq x} \sum_{d \mid n} \frac{\mu(d)^2}{\varphi(d)}$$

$$= \sum_{1 \leq d \leq x} \frac{\mu(d)^2}{\varphi(d)} \underbrace{\sum_{1 \leq m \leq x/d} 1}_{\lfloor x/d \rfloor}$$

(9)

$$= x \sum_{1 \le d \le x} \frac{\mu(d)^2}{d\varphi(d)} + O\left(\sum_{1 \le d \le x} \frac{1}{\phi(d)} \right)$$

(use, ex. $\varphi(d) \gg d^{1-\varepsilon}$)

$$= x \sum_{d=1}^{\infty} \frac{\mu(d)^2}{d\varphi(d)} + O(x^{\varepsilon}).$$

$\downarrow$

Use Euler product to evaluate this constant.

Check(?) : $$\sum_{d=1}^{\infty} \frac{\mu(d)^2}{d\varphi(d)} = \frac{\zeta(2)\zeta(3)}{\zeta(6)}.$$

7 Sep 2020 ①

Recall: arithmetic functions / averages

$f : \mathbb{N} \to \mathbb{C}$, find $g : \mathbb{N} \to \mathbb{C}$ s.t. $f(n) = \sum_{d|n} g(d)$

Then $$\frac{1}{x} \sum_{1 \le n \le x} f(n) = \frac{1}{x} \sum_{1 \le n \le x} \sum_{d|n} g(d) = \frac{1}{x} \sum_{1 \le d \le x} g(d) \sum_{m=1}^{x/d} 1$$

$$= \sum_{1 \le d \le x} \frac{g(d)}{d} + O\left(\frac{1}{x} \sum_{1 \le d \le x} |g(d)|\right).$$

Example 4.1: $$\frac{1}{x} \sum_{n \le x} \frac{\sigma(n)}{n} = \frac{\pi^2}{6} + O\left(\frac{\log x}{x}\right).$$ $\sigma(n) = \sum_{d|n} d$

Example 4.2: $$\frac{1}{x} \sum_{n \le x} \frac{n}{\phi(n)} = \frac{1}{x} \sum_{1 \le n \le x} \sum_{d|n} \frac{\mu(d)^2}{\varphi(d)} = \frac{1}{x} \sum_{1 \le d \le x} \frac{\mu(d)^2}{\varphi(d)} \left\lfloor \frac{x}{d} \right\rfloor$$

$$= \sum_{1 \le d \le x} \frac{\mu(d)^2}{d\varphi(d)} + O\left(\frac{1}{x} \sum_{1 \le d \le x} \frac{1}{\varphi(d)}\right).$$

Exercise: Show $\varphi(d) \gg_\varepsilon d^{1-\varepsilon}$ for any $\varepsilon > 0$.

Then $$\sum_{1 \le d \le x} \frac{1}{\varphi(d)} \ll x^\varepsilon \quad \text{and} \quad \sum_{d > x} \frac{\mu(d)^2}{d\varphi(d)} \ll x^{\varepsilon - 1}.$$

② Can work harder with these using the techniques from this section to show

$$\sum_{1\le d\le x} \frac{1}{\varphi(d)} \ll \log x \quad \text{and} \quad \sum_{d>x} \frac{\mu(d)^2}{d\,\varphi(d)} \ll x^{-1}.$$

Thus

$$\frac{1}{x}\sum_{n\le x} \frac{n}{\varphi(n)} = \underbrace{\sum_{d=1}^{\infty} \frac{\mu(d)^2}{d\,\varphi(d)}}_{\text{abs. conv. } = C, \text{ say.}} + O\left(\frac{\log x}{x}\right).$$

$O(x^{\varepsilon-1})$

Use multiplicative property of μ, φ, to show

$$\sum_{h=0}^{\infty} \frac{\mu(p^h)^2}{p^h\varphi(p^h)}$$

$$\sum_{d=1}^{\infty} \frac{\mu(d)^2}{d\,\varphi(d)} = \prod_p \left(1 + \frac{\mu(p)^2}{p(p-1)} + \frac{\mu(p^2)}{p^2\cdot p(p-1)} + \cdots\right)$$

$$= \prod_p \left(1 + \frac{1}{p(p-1)}\right) = \prod_p \left(\frac{p(p-1)}{p^2-p+1}\right)^{-1}$$

$$= \prod_p \left(\frac{p^3-1}{p^6-1} \cdot p(p-1)(p+1)\right)^{-1}$$

$$= \prod_p (1-p^{-6}) \cdot \prod_p (1-p^{-3})^{-1} \cdot \prod_p (1-p^{-2})^{-1}$$

$$= \zeta(2)\zeta(3)/\zeta(6).$$

③ Thus
$$\frac{1}{x}\sum_{1\le n\le x}\frac{n}{\varphi(n)} = \frac{\zeta(2)\zeta(3)}{\zeta(6)} + O(x^{\varepsilon-1}). \qquad \square$$

Variants of the basic idea:

(a) Powered convolutions: of shape $f(n) = \sum_{d^k \mid n} g(d)$ $(k>1)$

Example 4.3. (exercise) Show that $\mu(n)^2 = \sum_{d^2\mid n}\mu(d)$, and hence deduce that
$$\sum_{\substack{1\le n\le x\\ n \text{ squarefree}}} 1 = \frac{6}{\pi^2}x + O(x^{1/2}).$$

(b) Beyond Möbius inversion, I: rewrite the convolution sum in the shape
$$f(n) = \sum_{d\mid n} a(d)\, b(n/d),$$
for suitable arithmetic functions a, b. Then $(n = md)$
$$\sum_{1\le n\le x} f(n) = \sum_{1\le d\le x} a(d) \sum_{1\le m\le x/d} b(m)$$

④

$\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad ||$

$B(x/d)$, say — need a good estimate here.

<u>(c) Beyond Möbius inversion, II</u> : The "bilinear" structure can be used to improve error terms – idea is that if $n = k\ell$ then either $k \le \sqrt{n}$ or $\ell \le \sqrt{n}$ (or both).

More generally, suppose

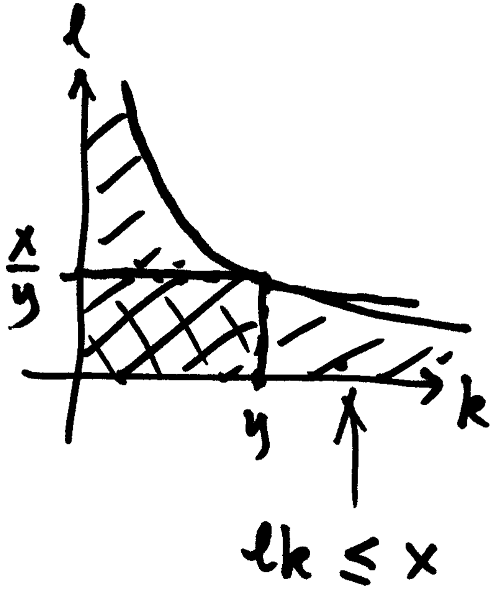
$$f(n) = \sum_{d \mid n} a(d)\, b(n/d) = \sum_{k\ell = n} a(k) b(\ell).$$

Write $\quad A(z) = \sum_{1 \le n \le z} a(n) \quad$ and $\quad B(z) = \sum_{1 \le n \le z} b(n)$.

Then if $1 \le y \le x$, (y a parameter)

$$\sum_{1 \le n \le x} f(n) = \sum_{1 \le k\ell \le x} a(k) b(\ell)$$

$$= \sum_{1 \le k \le y} \sum_{1 \le \ell \le x/k} a(k) b(\ell) +$$



$$+ \sum_{1\le l\le x/y} \; \sum_{1\le k\le x/l} a(k)\, b(l)$$

$$- \sum_{1\le k\le y} \; \sum_{1\le l\le x/y} a(k)\, b(l)$$

"overcounting".

$$= \sum_{1\le k\le y}{}' \; a(k)\, B(x/k)$$

$$+ \sum_{1\le l\le x/y} b(l)\, A(x/l) - A(y)\, B(x/y).$$

So

$$\sum_{1\le n\le x}{}' f(n) = \sum_{1\le k\le y} a(k) B(x/k) + \sum_{1\le l\le x/y} b(l) A(x/l)$$

$$- A(y) B(x/y).$$

⑥ <u>Example 4.4</u>: Observe, naively, we have

$$\sum_{1\le n\le x} \tau(n) = \sum_{1\le d\le x} \sum_{1\le m\le x/d} 1 = \sum_{1\le d\le x} \left\lfloor \frac{x}{d} \right\rfloor$$

$$\sum_{1\le n\le x} \tau(n) = \sum_{1\le n\le x} \sum_{d\mid n} 1$$

$$\sum_{1\le d\le x} \left\lfloor \frac{x}{d} \right\rfloor = x \sum_{1\le d\le x} \frac{1}{d} + O(x) = x\log x + O(x).$$

Refined strategy from (c):

$$\sum_{1\le n\le x} \tau(n) = \sum_{1\le kl\le x} 1 = \sum_{1\le k\le y} \underbrace{\left\lfloor \frac{x}{k} \right\rfloor}_{\left(\sum_{1\le l\le x/k} 1\right)} + \sum_{1\le l\le x/y} \left\lfloor \frac{x}{l} \right\rfloor - \lfloor y\rfloor \lfloor x/y \rfloor$$

$$= x\sum_{1\le k\le y} \frac{1}{k} + x\sum_{1\le l\le x/y} 1/l - y\cdot x/y + O(y + x/y)$$

$$= x\left(\log y + C_0 + O\left(\tfrac{1}{y}\right)\right) + x\left(\log\left(\tfrac{x}{y}\right) + C_0 + O\left(\tfrac{1}{x/y}\right)\right)$$

$$- x + O\left(y + x/y\right)$$

$$\sum_{1\le n\le x} \tau(n) = x\log x + (2C_0 - 1)x + O(y + x/y).$$

By taking $y = \sqrt{x}$ to get error term $O(\sqrt{x})$.

§5. Prime number theorems: the work of Chebyshev & Mertens.

Some notation

$$\pi(x) = \sum_{p\le x} 1, \quad \theta(x) = \sum_{p\le x}\log p, \quad \psi(x) = \sum_{n\le x}\Lambda(n).$$

Theorem 5.1. When $x \ge 2$, one has $\psi(x) \asymp x$.

(i.e. $\exists\, c_1, c_2 > 0$ s.t. $c_1 x < \psi(x) < c_2 x$).

Proof. Idea – use detector for prime powers, say

$$\sum_{1 \le n \le x} \sum_{d \mid n} \mu(d) = \begin{cases} 1, & n \in [1, x] \\ 0, & \text{otherwise} \end{cases}$$

$$\psi(x) = \sum_{1 \le n \le x} \Lambda(n) = \sum_{1 \le n \le x} \left(\sum_{1 \le m \le x/n} \sum_{d \mid m} \mu(d) \right) \Lambda(n)$$

[But try to insert a better behaved – smoothed out – function which is majorised by this weight]

Substitute for Möbius function, say $a(d)$. Suggest

$$a(d) = \begin{cases} 1, & \text{when } d = 1, \\ -2, & \text{when } d = 2, \\ 0, & \text{when } d \ge 3. \end{cases}$$

We then have

$m = ky$

$$\sum_{1 \le m \le y} \sum_{d \mid m} a(d) = \sum_{1 \le d \le y} a(d) \sum_{1 \le k \le y/d} 1$$

$\lfloor y/d \rfloor$

$$= \lfloor y \rfloor - 2\lfloor y/2 \rfloor \in \{0, 1\}$$

We deduce that

$$\sum_{1 \leq n \leq x} \lambda(n) \left(\left\lfloor \frac{x}{n} \right\rfloor - 2\left\lfloor \frac{x}{2n} \right\rfloor \right) \leq \psi(x)$$

Since for $x \geq n > x/2$, the parenthetic expression is equal to 1, so

$$\psi(x) - \psi(x/2) \leq \sum_{1 \leq n \leq x} \lambda(n) \sum_{1 \leq m \leq x/n} \sum_{d \mid m} a(d) \leq \psi(x).$$

9 Sep 2020

①

Recall: Prime Number Theorems – Chebyshev & Mertens

$$\pi(x) = \sum_{p \leq x} 1, \qquad \theta(x) = \sum_{p \leq x} \log p, \qquad \psi(x) = \sum_{n \leq x} \Lambda(n)$$

Theorem 5.1 One has $\psi(x) \asymp x$ for $x \geq 2$.

(means $\exists\, c_1, c_2 > 0$ s.t. $c_1 x \leq \psi(x) \leq c_2 x$).

Proof. (so far). Motivated by

$$\sum_{1 \leq m \leq y} \sum_{d \mid m} \mu(d) = 1,$$

choose substitute for $\mu(d)$, say $a(d)$, with $\sum_{1 \leq m \leq y} \sum_{d \mid m} a(d) \in (0,1)$.

For example
$$a(d) = \begin{cases} 1, & \text{when } d = 1, \\ -2, & \text{when } d = 2, \\ 0, & \text{when } d \geq 3. \end{cases}$$

Then
$$\sum_{1 \leq m \leq y} \sum_{d \mid m} a(d) = \lfloor y \rfloor - 2 \lfloor y/2 \rfloor \in \{0, 1\},$$

and
$$\psi(x) - \psi(x/2) \leq \sum_{1 \leq n \leq x} \Lambda(n) \sum_{1 \leq m \leq x/n} \sum_{d \mid m} a(d) \leq \psi(x)$$

②

$$\sum_{1\le kdn\le x}'' \Lambda(n)a(d) = \sum_{1\le \ell d\le x} a(d) \underbrace{\sum_{n\mid \ell}\Lambda(n)}_{= \log \ell}$$

$$\ell = kn$$

$$= \sum_{1\le \ell d\le x}' a(d)\log \ell = \sum_{1\le d\le x} a(d)\left(\sum_{1\le \ell\le x/d}\log\ell\right)$$

By R-S integration, one has

$$\sum_{1\le \ell\le y}\log \ell = \int_{1^-}^{y} \log u \, d\lfloor u\rfloor = y\log y - y + O(\log y)$$

(✓)

Thus

$d=1$

$$\sum_{1\le d\le x} a(d)\sum_{1\le \ell\le x/d}\log \ell = \left(x\log x - x + O(\log x)\right) - 2\left(\frac{x}{2}\log\left(\frac{x}{2}\right) - \frac{x}{2} + O(\log x)\right)$$

$d=2$

③

$$= x\log 2 + O(\log x)$$

Thus $\psi(x) \geq x \log 2 + O(\log x)$

and

$$\psi(x) - \psi(x/2) \leq x(\log 2) + O(\log x)$$
$$\psi(x/2) - \psi(x/4) \leq \frac{x}{2}(\log 2) + O(\log x)$$
$$\vdots$$

Summing over dyadic int. $\longrightarrow$ $\psi(x) \leq 2x(\log 2) + O((\log x)^2)$.

Thus

$$(\log 2)x + O(\log x) \leq \psi(x) \leq (2\log 2)x + O((\log x)^2).$$

Chebyshev : $a(1) = a(30) = 1$
$a(2) = a(3) = a(5) = -1$
$a(d) = 0$, otherwise.

$$\sum_{d|n} \Lambda(d) = \log n. \qquad -\frac{\zeta'}{\zeta}(s)\,\zeta(s) = -\zeta'(s)$$

④ $$0.9212\,x + O(\log x) \leq \psi(x) \leq 1.1056\,x + O((\log x)^2)$$

<u>Corollary 5.2</u> When $x \geq 2$, one has

$$\theta(x) = \psi(x) + O(x^{1/2}) \quad \text{and} \quad \pi(x) = \frac{\psi(x)}{\log x} + O\left(\frac{x}{(\log x)^2}\right).$$

Moreover, one has $\theta(x) \asymp x$ and $\pi(x) \asymp x/\log x$.

<u>Proof.</u> By definitions of ψ & θ, we have

$$\psi(x) = \sum_{p^k \leq x} \log p = \sum_{k=1}^{\infty} \theta(x^{1/k}).$$

But for $y \in \mathbb{R}_{\geq 2}$, one has $\theta(y) \leq \psi(y) \asymp y$,

So $$\psi(x) - \theta(x) = \sum_{k \geq 2} \theta(x^{1/k}) \ll x^{\frac{1}{2}} + x^{1/3} \log x$$

$$\ll x^{1/2}. \quad \square \qquad \Rightarrow \theta(x) = \psi(x) + O(x^{\frac{1}{2}}) \asymp x.$$

(5)

We apply R-S integration:

$$\pi(x) = \int_{2^-}^{x} \frac{1}{\log u}\, d\,\theta(u) = \frac{\theta(x)}{\log x} + \int_2^x \frac{\theta(u)}{u(\log u)^2}\, du$$

$$= \frac{\theta(x)}{\log x} + O\left(\int_2^x \frac{du}{(\log u)^2} \right) \simeq \frac{x}{\log x} + O\left(\frac{x}{(\log x)^2}\right).$$

Hence $$\pi(x) = \frac{\psi(x)}{\log x} + O\left(\frac{x}{(\log x)^2}\right) \simeq \frac{x}{\log x}.$$

//

⑥

Theorem 5.3. When $x \geq 2$, one has

(a) $$\sum_{n \leq x} \frac{\Lambda(n)}{n} = \log x + O(1) \quad \text{and} \quad \sum_{p \leq x} \frac{\log p}{p} = \log x + O(1);$$

(b) $$\int_1^x \frac{\psi(u)}{u^2}\, du = \log x + O(1);$$

(c) $$\sum_{p \leq x} \frac{1}{p} = \log\log x + C_0 - \sum_p \sum_{k=2}^{\infty} \frac{1}{kp^k} + O\left(\frac{1}{\log x}\right);$$

(d) $$\prod_{p \leq x} (1 - 1/p)^{-1} = e^{C_0} \log x + O(1) \quad \text{(Mertens' 3rd Thm)}$$

Proof: (a) We have $\log n = \sum_{d \mid n} \Lambda(d)$, and hence

$$\sum_{1 \leq n \leq x} \log n = \sum_{1 \leq d \leq x} \Lambda(d) \sum_{1 \leq m \leq x/d} 1 = x \sum_{1 \leq d \leq x} \frac{\Lambda(d)}{d} + O(\psi(x))$$

⑦

$$\Rightarrow \sum_{1\le d\le x} \frac{\Lambda(d)}{d} = \frac{x\log x - x + O(\log x)}{x} + \frac{O(x)}{x}$$

$$= \log x + O(1).$$

Moreover, we have

$$\sum_{1\le d\le x} \frac{\Lambda(d)}{d} = \sum_{p\le x} \frac{\log p}{p} + \sum_{k=2}^{\infty} \sum_{p^k\le x} \frac{\log p}{p^k}$$

$$\log x + O(1) = \sum_{p\le x} \frac{\log p}{p} + \sum_{p} \frac{\log p}{p(p-1)} = \sum_{p\le x} \frac{\log p}{p} + O(1).$$

(b) We have

$$\sum_{1\le d\le x} \frac{\Lambda(d)}{d} = \int_{2^-}^{x} u^{-1}\, d\psi(u) = \frac{\psi(u)}{u}\Big|_{2^-}^{x} + \int_{2}^{x} \frac{\psi(u)}{u^2}\, du$$

$$= \int_{1}^{x} \frac{\psi(u)}{u^2}\, du + O(1). \quad \square$$

11 Sep 2020

1

Recall: Theorem 5.3 When $x \geqslant 2$, one has

✓ (a) $\displaystyle\sum_{n \leq x} \frac{\Lambda(n)}{n} = \log x + O(1)$ and $\displaystyle\sum_{p \leq x} \frac{\log p}{p} = \log x + O(1)$;

✓ (b) $\displaystyle\int_1^x \frac{\psi(u)}{u^2}\, du = \log x + O(1)$;

(c) $\displaystyle\sum_{p \leq x} \frac{1}{p} = \log\log x + C_0 - \sum_p \sum_{k=2}^{\infty} \frac{1}{kp^k} + O\left(\frac{1}{\log x}\right)$;

(d) $\displaystyle\prod_{p \leq x} \left(1 - \frac{1}{p}\right)^{-1} = e^{C_0} \log x + O(1)$ (Mertens' 3rd Theorem

$\boxed{\psi(x) \simeq x}$

$$\sum_{d=1}^{\infty} \frac{\mu(d)}{d} = \prod_p \left(1 - \frac{1}{p}\right) = 0.$$

Proof (c) & (d): Define

$$L(x) = \sum_{p \leq x} \frac{\log p}{p} \quad \text{and} \quad R(x) = L(x) - \log x,$$

so ~~that~~ by part (a), we have $R(x) \ll 1$. Then by R–S integration,

②

$$\sum_{p\le x}\frac{1}{p} = \int_{2^-}^{x}\frac{1}{\log u}\,dL(u) = \int_{2^-}^{x}\frac{d(\log u)}{\log u} + \int_{2^-}^{x}\frac{dR(u)}{\log u}$$

$$= \log\log u\Big|_{2^-}^{x} + \frac{R(u)}{\log u}\Big|_{2^-}^{x} + \int_{2^-}^{x}\frac{R(u)}{u(\log u)^2}\,du$$

$$= \log\log x - \log\log 2 + \frac{R(x)}{\log x} + \underbrace{1}_{R(2^-)} + \int_{2}^{\infty}\frac{R(u)}{u(\log u)^2}\,du$$

$$- \int_{x}^{\infty}\frac{R(u)}{u(\log u)^2}\,du$$

Write $b = 1 - \log\log 2 + \int_2^{\infty}\frac{R(u)}{u(\log u)^2}\,du$, and recall that $R(u) \ll 1$. Thus

$$\sum_{p\le x}\frac{1}{p} = \log\log x + b + O\left(\frac{1}{\log x}\right). \quad (*) \quad (c)$$

③ (d) Note ~~that~~

$$\log\Big(\prod_{p\leq x}\Big(1-\frac{1}{p}\Big)^{-1}\Big) = -\sum_{p\leq x}\log\Big(1-\frac{1}{p}\Big)$$

$$= \sum_{p\leq x}\frac{1}{p} + \sum_{p\leq x}\sum_{k=2}^{\infty}\frac{1}{kp^k}.$$

By part (c), we have

$$\log\Big(\prod_{p\leq x}\Big(1-\frac{1}{p}\Big)^{-1}\Big) = \log\log x + c - \underbrace{\sum_{p>x}\sum_{k=2}^{\infty}\frac{1}{kp^k}}_{\leq \sum_{n>x}\frac{1}{n^2} = O\left(\frac{1}{x}\right)}, \quad + O\Big(\frac{1}{\log x}\Big)$$

where

$$c = b + \sum_{p}\sum_{k=2}^{\infty}\frac{1}{kp^k}.$$

Hence

$$\prod_{p\leq x}\Big(1-\frac{1}{p}\Big)^{-1} = \big(e^{c}\log x\big)\Big(1+O\Big(\frac{1}{\log x}\Big)\Big)$$

$$= e^{c}\log x + O(1).$$

$\circledast$ (d)

(c) + (d) Prove $c = C_0$. We begin by observing ~~that~~

$$\sum_{1 \le n \le x} \frac{\Lambda(n)}{n \log n} = \sum_{p \le x} \sum_{k=1}^{\infty} \frac{\Lambda(p^k)}{p^k \log(p^k)} - \sum_{\substack{p \le x \\ p^k > x}} \frac{\Lambda(p^k)}{p^k \log(p^k)}$$

$$= -\sum_{p \le x} \log\left(1 - \frac{1}{p}\right) + O\left(\frac{1}{\log x} \sum_{p \le x} \sum_{k \ge 2} \frac{\log p}{p^k}\right)$$

$$= \log\Big(\underbrace{\prod_{p \le x} \left(1 - 1/p\right)^{-1}}_{e^c \log x + O(1)} \Big) + O\left(1/\log(2x) \right).$$

Hence (using (3.4)), we deduce ~~that~~

$$\sum_{1 < n \le x} \frac{\Lambda(n)}{n \log n} = \log\log x + c + O\left(\frac{1}{\log(2x)}\right)$$

$$= \sum_{1 \le n \le \log x} \frac{1}{n} + (c - C_0) + O\left(\frac{1}{\log(2x)}\right). \qquad (5.1)$$

We next introduce auxiliary quantities, for $\delta > 0$:

⑤

$$I_1(\delta) = \delta \int_{1^+}^{\infty} x^{-1-\delta} \sum_{1<n\le x} \frac{\Lambda(n)}{n\log n}\, dx,$$

$$I_2(\delta) = \delta \int_{1^+}^{\infty} x^{-1-\delta} \sum_{1\le n\le \log x} \frac{1}{n}\, dx,$$

$$I_3(\delta) = \delta \int_{1^+}^{\infty} x^{-1-\delta}\, dx = 1.$$

Then it follows from (5.1) that

$$I_1(\delta) = I_2(\delta) + c - C_0 + I_4(\delta),$$

where

$$I_4(\delta) \ll \delta \int_{1^+}^{\infty} \frac{x^{-1-\delta}}{\log(2x)}\, dx \ll \delta + \delta \int_2^{e^{1/\delta}} \frac{dx}{x\log x} + \delta^2 \int_{e^{1/\delta}}^{\infty} x^{-1-\delta}\, dx$$

$$\ll \delta \log(1/\delta). \qquad (\delta \to 0+)$$

But by Theorem 2.4 (iii) have

⑥ $$I_1(\delta) = \sum_{n=1}^{\infty} a_n n^{-\delta}, \quad \text{where } a_n = \Lambda(n)/(n\log n),$$

whence $$I_1(\delta) = \log\zeta(1+\delta) = \log(1/\delta) + O(\delta),$$
as $\delta \to 0+$.

(using $\zeta(s) = \frac{1}{s-1} + c_0 + O(|s-1|)$). Also,

$$I_2(\delta) = \delta \sum_{n=1}^{\infty} \frac{1}{n} \int_{e^n}^{\infty} x^{-1-\delta}\,dx = \sum_{n=1}^{\infty} \frac{1}{n} e^{-\delta n}$$

$$= -\log(1-e^{-\delta}) = -\log(\delta + O(\delta^2))$$

$$= \log(1/\delta) + O(\delta) \quad \text{as } \delta \to 0+.$$

We therefore conclude that as $\delta \to 0+$, one has

$$\log\left(\frac{1}{\delta}\right) = \log\left(\frac{1}{\delta}\right) + c - c_0 + \underbrace{O(\delta\log(1/\delta))}_{\to 0 \text{ as } \delta\to 0+}$$

⑦

So $\quad c = C_0 \quad$ (by taking $\delta \to 0+$).

Thus
$$\prod_{p \le x} \left(1 - \frac{1}{p}\right)^{-1} = e^{C_0} \log x + O(1),$$

and
$$b = C_0 - \sum_{p} \sum_{k=2}^{\infty} \frac{1}{k p^k}.$$

$$\sum_{n \le x} \frac{1}{n} = \log x + C_0 + O\left(\frac{1}{x}\right)$$

14 Sep 2020 | Recall: **Theorem 5.3** When $x \geq 2$, one has

①

(a) $$\sum_{n \leq x} \frac{\Lambda(n)}{n} = \log x + O(1) \quad \text{and} \quad \sum_{p \leq x} \frac{\log p}{p} = \log x + O(1);$$

→ (b) $$\int_1^x \frac{\psi(u)}{u^2}\, du = \log x + O(1);$$

(c) $$\sum_{p \leq x} \frac{1}{p} = \log\log x + \underbrace{C_0 - \sum_p \sum_{k=2}^{\infty} \frac{1}{kp^k}}_{b} + O(1/\log x);$$

(d) $$\prod_{p \leq x} (1 - 1/p)^{-1} = e^{C_0} \log x + O(1).$$ (Mertens' 3rd Theorem)

Corollary 5.4. One has

$$\liminf_{x \to \infty} \frac{\pi(x)}{x/\log x} \leq 1 \quad \text{and} \quad \limsup_{x \to \infty} \frac{\pi(x)}{x/\log x} \geq 1.$$

[Recall Corollary 5.2 : $\pi(x) = \dfrac{\psi(x)}{\log x} + O\left(\dfrac{x}{(\log x)^2}\right)$.]

$$\psi(x) = \sum_{n \leq x} \Lambda(n)$$

Proof: Note ~~that~~ it suffices to prove ~~that~~

$$\liminf_{x \to \infty} \frac{\psi(x)}{x} \leq 1 \quad \text{and} \quad \limsup_{x \to \infty} \frac{\psi(x)}{x} \geq 1.$$

(2) Suppose, if possible, ~~that~~ $\limsup_{x\to\infty} \frac{\psi(x)}{x} = a$

$\longrightarrow$ * [Have $\psi(x) \asymp x$ for all x].

(have in mind $\overset{?}{a} \overset{?}{\neq} 1$). Then given $\varepsilon > 0$ one must have $\psi(x) \leq (a+\varepsilon)x$ for all large enough x, say $x \geq x_0(\varepsilon)$. But then

$$\int_1^X \frac{\psi(u)}{u^2}\,du \leq \int_1^{x_0} \frac{\psi(u)}{u^2}\,du + (a+\varepsilon)\int_{x_0}^X u^{-1}\,du$$

$$\leq (a+\varepsilon)\log(X/x_0) + O_\varepsilon(1).$$

$$= (a+\varepsilon)\log x + O_\varepsilon(1).$$

Thus $\int_1^x \frac{\psi(u)}{u^2}\,du \leq (a+o(1))\log x$.

But by Theorem 5.3, have $\int_1^x \frac{\psi(u)}{u^2}\,du = \log x + O(1)$, whence $a \geq 1$. Then

$$\limsup_{x\to\infty} \frac{\psi(x)}{x} \geq 1. \quad \square.$$

Similar argument for $\liminf_{x\to\infty} \frac{\psi(x)}{x} \leq 1$. //

§6. Bounds for arithmetic functions.

Theorem 6.1 When $n \geqslant 3$, one has

$$\varphi(n) \geqslant \frac{n}{\log\log n}\left(e^{-C_0} + O\left(1/\log\log n\right)\right);$$

(Euler's totient: $\varphi(n) = n\prod_{p|n}(1-1/p)$)

and

$$1 \leq \omega(n) \leq \frac{\log n}{\log\log n}\left(1 + O\left(\frac{1}{\log\log n}\right)\right).$$

$\left(\omega(n) = \sum_{p|n} 1\right)$

Proof. Denote the k-th largest prime by p_k, so that $p_1 = 2$, $p_2 = 3, \dots$, and put $m = \omega(n)$. Then since

$$\frac{\varphi(n)}{n} = \prod_{p|n}(1-1/p) \geqslant \prod_{i=1}^{m}\left(1 - 1/p_i\right),$$

we find from Mertens' theorem ~~that~~

$$\frac{\varphi(n)}{n} \geqslant \frac{e^{-C_0}}{\log p_m} + O(1).$$

④ However, we have $p_1 \cdots p_m \le \prod_{p \mid n} p \le n$,

whence

$$\underbrace{\sum_{p \le p_m} \log p}_{\substack{\parallel \\ \theta(p_m) \,\simeq\, p_m + O(1)}} \le \log n \quad \Longrightarrow \quad \log p_m \le \log\log n + O(1).$$

$\boxed{\begin{array}{l} \psi(x) \simeq x \leftrightarrow \theta(x) = \psi(x) + O(x^{\frac{1}{2}}) \\ \pi(x) = \theta(x)/\log x + \text{error}. \end{array}}$

$$\Rightarrow \quad \frac{\varphi(n)}{n} \geq \frac{e^{-c_0}}{\log\log n}\left(1 + O\left(\frac{1}{\log\log n}\right)\right). \quad \square$$

Likewise, we have $\omega(n) \le m$, where m is the largest natural number with $\underbrace{p_1 \cdots p_m}_{\substack{\parallel \\ n^*,\ \text{say}}} \le n$.

Then

$$\sum_{p \le p_m} \log p = \log n^* \quad \Longrightarrow \quad \log p_m = \log\log n^* + O(1)$$

$$\simeq \theta(p_m) \simeq p_m + O(1)$$

Also

$$m = \pi(p_m) = \frac{\theta(p_m)}{\log p_m} + O\left(\frac{p_m}{(\log p_m)^2}\right)$$

$$= \frac{\log n^*}{\log\log n^* + O(1)} + O\left(\frac{\log n^*}{(\log\log n^*)^2}\right).$$

$n^* \le n$

$$\Rightarrow \omega(n) = m \le \frac{\log n}{\log\log n}\left(1 + O\left(\frac{1}{\log\log n}\right)\right). \quad \square$$

<u>Ex.</u> $\tau(n) = \sum_{d \mid n} 1$ satisfies: $\log \tau(n) \le \left(\frac{\log n}{\log\log n}\right) \times \log 2 \, (1+o(1))$.

<u>Theorem 6.2</u> (Turán) For $x \ge 3$, one has

$$\sum_{n \le x} (\omega(n) - \log\log n)^2 \ll x \log\log x . \qquad (6.1)$$

⑥

<u>Proof.</u> First show that (6.1) holds with $\log\log x$ in place of $\log\log n$.

Indeed, one has

$$\sum_{n\le x}(\omega(n)-\log\log x)^2 = S_2 - 2(\log\log x)S_1 + \lfloor x\rfloor(\log\log x)^2, \quad (6.2)$$

where

$$S_j = \sum_{n\le x}\omega(n)^j \qquad (j=1,2).$$

But

$$S_1 = \sum_{n\le x}\omega(n) = \sum_{n\le x}\sum_{p\mid n}1$$

$$= \sum_{p\le x}\sum_{1\le m\le x/p}1 = \sum_{p\le x}\left\lfloor\frac{x}{p}\right\rfloor$$

$$= x\sum_{p\le x}\frac{1}{p} + O\Big(\sum_{p\le x}1\Big).$$

(7)

$$\longrightarrow \quad S_1 = x(\log\log x + b) + O(x/\log x).$$

Also,

$$S_2 = \sum_{n \le x} \Big(\sum_{p_1 | n} 1 \Big) \Big(\sum_{p_2 | n} 1 \Big) = \sum_{p_1 \le x} \sum_{p_2 \le x} \sum_{\substack{1 \le n \le x \\ p_1 | n \\ \& \ p_2 | n}} 1$$

Cont.^{n} from $p_1 = p_2$:

$$\sum_{p \le x} \left\lfloor \frac{x}{p} \right\rfloor = x \log\log x + O(x)$$

Cont.^{n} from $p_1 \ne p_2$:

$$\sum_{p_1 \le x} \sum_{\substack{p_2 \le x \\ p_2 \ne p_1}} \left\lfloor \frac{x}{p_1 p_2} \right\rfloor \le x \Big(\sum_{p \le x} \frac{1}{p} \Big)^2 = x(\log\log x)^2 + O(x \log\log x)$$

Hence $\quad S_2 \le x(\log\log x)^2 + O(x \log\log x)$.

⑧ Sub. into (6.2):

$$\sum_{n \le x} (\omega(n) - \log\log x)^2 \le x(\log\log x)^2 - 2(\log\log x) \times \log\log x + x(\log\log x)^2 + O(x\log\log x)$$

$$\ll x \log\log x. \quad \square.$$

Ex. Replace $\log\log x$ by $\log\log n$. //

16 Sep 2020

①

Recall: Bounds for arithmetic functions

- $n \geq \varphi(n) \geq \dfrac{n}{\log\log n}\left(e^{-c_0} + O\left(\dfrac{1}{\log\log n}\right)\right)$

- $1 \leq \omega(n) \leq \dfrac{\log n}{\log\log n}\left(1 + O\left(\dfrac{1}{\log\log n}\right)\right)$

  - But $\omega(n)$ is "typically" $\approx \log\log n$, since
  $$\sum_{n \leq x} (\omega(n) - \log\log n)^2 \ll x \log\log x$$

- (Ex?) $\log(\tau(n)) \leq \dfrac{\log n}{\log\log n}(\log 2 + o(1))$
$$\Rightarrow \tau(n) \ll 2^{(1+o(1))\log n / \log\log n}$$
$$\ll_\varepsilon n^\varepsilon \quad (\text{any } \varepsilon > 0).$$

§7. Additive and multiplicative characters.

Goal over next few classes:

Theorem 7.1. (Dirichlet, 1837) Suppose that $a \in \mathbb{Z}$, $q \in \mathbb{N}$ and $(a, q) = 1$. Then there are infinitely many primes p with $p \equiv a \pmod q$.

Detect $a \pmod q$ $\longleftrightarrow$ Fourier analysis on $(\mathbb{Z}/q\mathbb{Z})^{+}$

$\longleftrightarrow$ Fourier analysis on $(\mathbb{Z}/q\mathbb{Z})^{\times}$.
"reduced residues" $(a, q) = 1$.

Additive characters:

$$e(\theta) := e^{2\pi i \theta}$$

$$e_q(a) := e(a/q) \qquad (a \in \mathbb{Z},\ q \in \mathbb{N})$$

Orthogonality: One has

$$q^{-1} \sum_{a=1}^{q} e(ah/q) = \begin{cases} 1, & \text{when } h \equiv 0 \pmod q, \\ 0, & \text{when } h \not\equiv 0 \pmod q. \end{cases}$$

3 [If $q \nmid h$, have $\displaystyle\sum_{a=1}^{q} e(ah/q) = e(h/q)\left(\frac{e(h)-1}{e(h/q)-1}\right) = 0.$]

Claim: Given $f: \mathbb{Z} \longrightarrow \mathbb{C}$ periodic, period q, then f has expansion.

$$\hat{f}(k) := \frac{1}{q}\sum_{n=1}^{q} f(n)\, e(-kn/q). \qquad \text{——— (7.1)}$$

Then

$$f(n) = \sum_{k=1}^{q} \hat{f}(k)\, e(kn/q). \qquad \text{——— (7.2)}$$

Proof: By orthogonality,

$$\sum_{k=1}^{q} \hat{f}(k) e(kn/q) = q^{-1}\sum_{k=1}^{q}\left(\sum_{m=1}^{q} f(m)\, e(-km/q)\right) e(kn/q)$$

$$= \sum_{m=1}^{q} \underbrace{\left(q^{-1}\sum_{k=1}^{q} e(k(n-m)/q)\right)}_{\begin{cases} =0, & \text{when } n \not\equiv m \pmod q \\ =1, & \text{when } n \equiv m \pmod q \end{cases}} f(m)$$

$\|$

$f(n)$. ✓

④ **Note:** characters $e(kn/q)$ $(n \in \mathbb{Z})$ are indeed additive for $k \in \mathbb{Z}$, in the sense

$$e(k_1 n/q)\, e(k_2 n/q) = e((k_1+k_2)n/q).$$

Analogue of Parseval/Plancharel: (Ex.)

$$\sum_{k=1}^{q} |\hat{f}(k)|^2 = q^{-1} \sum_{n=1}^{q} |f(n)|^2. \qquad (7.3)$$

Multiplicative characters: Interested in $f: \mathbb{Z} \to \mathbb{C}$ having the property ~~that~~ f is supported on $n \in \mathbb{Z}$ with $(n,q)=1$, and f is periodic with periods q.

$(\mathbb{Z}/q\mathbb{Z})^\times$.

5

Definition 7.1. A Dirichlet character is a totally multiplicative function $\chi : \mathbb{Z} \to \mathbb{C}$ having the properties:

(i) $\chi(n) = 0$ if and only if $(n,q) > 1$;

and (ii) χ is periodic with period q.

Theorem 7.2 Suppose that $f : \mathbb{N} \to \mathbb{C}$ is multiplicative, and satisfies:

(i) $f(n) = 0$ whenever $(n,q) > 1$,

and (ii) f is periodic with period q.

Then f is a Dirichlet character modulo q.

Proof: It suffices to show f is totally mult., with property that $f(n) \neq 0$ whenever $(n,q) = 1$.

⑥ First, when $(mn,q)>1$, then either $(m,q)>1$ or $(n,q)>1$, so

$$f(m)f(n) = 0 = f(mn). \checkmark$$

Suppose then $(mn,q)=1$, so $(m,q)=(n,q)=1$. Then there exists $k \in \mathbb{Z}$ s.t. $n+kq \equiv 1 \pmod{m}$, so that $(m, n+kq) = 1$. Then mult. property of f ensures that

$$\begin{array}{ccc} f(m(n+kq)) & = & f(m)f(n+kq) \\ \text{periodicity} \quad \| & & \| \\ f(mn) & & f(m)f(n) \end{array}$$

Thus f is totally multiplicative $\square$.

To check ~~that~~ $f(n) \neq 0$ whenever $(n,q)=1$, observe that $f(n)^{\varphi(q)} = f(n^{\varphi(q)}) = f(1) = 1$, whence $f(n)$ is a $\varphi(q)$-th root of unity, hence $\neq 0$. //

⑦ **Remarks:** $\chi : \mathbb{Z} \longrightarrow \mathbb{C}$

We can think of χ mod q as a map

$$\chi : (\mathbb{Z}/q\mathbb{Z})^\times \longrightarrow \mathbb{C}^\times .$$

Since $\chi(mn) = \chi(m)\chi(n)$ all m, n, it follows ~~that~~ $\chi : (\mathbb{Z}/q\mathbb{Z})^\times \longrightarrow \mathbb{C}^\times$ is a group homomorphism.

Group of Dirichlet characters mod q:

- identity: χ_0 (principal character) defined by taking

$$\chi_0(n) = 1 \quad \text{whenever} \quad (n,q)=1$$

- inverses: given χ, the conjugate $\bar{\chi}$ defined by taking $\bar{\chi}(n) = \overline{\chi(n)}$ for $(n,q)=1$.

[$\bar{\chi}\chi(n) = \bar{\chi}(n)\chi(n) = 1 = \chi_0$ for $(n,q)=1$].

Recall: $\chi(n)^{\varphi(q)} = 1$, so $\chi(n)$ is a root of unity.

Example 7.3: Let q be a prime number, say p, and consider a character χ modulo p.

Note that $(\mathbb{Z}/p\mathbb{Z})^\times$ is a cyclic group of order $\varphi(p) = p-1$ having a generator g (a primitive root). Thus, given an integer n with $(n,p)=1$, ~~there~~ is an integer $u = \mathrm{ind}_g(n)$ having the property ~~that~~

$$n \equiv g^u \pmod{p}.$$

We then have

$$\chi(n) = \chi(g^u) = \chi(g)^u.$$

So the character χ is determined by its value at a primitive root g modulo p. This value is

9

a $(p-1)$-th root of unity.

It follows that there are precisely $p-1$ characters modulo p, namely

$$\chi_k : (\mathbb{Z}/p\mathbb{Z})^\times \longrightarrow \mathbb{C}^\times,$$

defined by taking

$$\chi_k(n) = e\left(\frac{k \operatorname{ind}_g(n)}{p-1}\right) \qquad (0 \leq k \leq p-2)$$

Since the values $\chi_k(g) = e\left(\frac{k}{p-1}\right)$ are distinct for $0 \leq k \leq p-2$, these Dirichlet characters are all distinct.

<u>Notice:</u> group of characters $\{\chi_0, \ldots, \chi_{p-2}\}$

$$= \{\chi_1^k : 0 \leq k \leq p-2\} \cong C_{p-1} \cong (\mathbb{Z}/p\mathbb{Z})^\times.$$

18 Sep 2020 | **Recall:**

①

Additive character (mod q) $\quad f : \mathbb{Z} \to \mathbb{C}$, periodic mod q

$$n \longmapsto e(hn/q) \qquad (0 \leq h < q)$$

Multiplicative character (mod q) $\quad f : \mathbb{Z} \longrightarrow \mathbb{C}$, periodic mod q

- Totally multiplicative, so $f(mn) = f(m)f(n)$, all m, n;
- $f(n) = 0$ if and only if $(n, q) > 1$.

(Can restrict $f : (\mathbb{Z}/q\mathbb{Z})^\times \longrightarrow \mathbb{C}^\times$)

Example 7.3 Characters χ modulo p

Let g be any primitive root modulo p, so $\langle g \rangle = (\mathbb{Z}/p\mathbb{Z})^\times$.

The characters χ modulo p are given by

$$\chi_k : (\mathbb{Z}/p\mathbb{Z})^\times \longrightarrow \mathbb{C}^\times \qquad \boxed{g^{\mathrm{ind}_g(n)} \equiv n \pmod{p}}$$

$$\chi_k(n) = e\left(\frac{k \, \mathrm{ind}_g(n)}{p-1}\right) \qquad (0 \leq k \leq p-2)$$

$$X(p) := \{\chi_1^k : 0 \leq k \leq p-2\} \cong C_{p-1} \cong (\mathbb{Z}/p\mathbb{Z})^\times.$$

$$\chi_0 = \text{principal} = \mathbb{1}_{(\mathbb{Z}/p\mathbb{Z})^\times}.$$

② **Example 7.4.** Let q be an odd prime power, say $q = p^h$, or else be 2 or 4, and consider the characters χ mod q.

Here $(\mathbb{Z}/p^h\mathbb{Z})^\times$ (with p odd, or $p^h \in \{2,4\}$) is again cyclic of order $\varphi(p^h)$, having a generator g (a primitive root).

(**Fact**: if g is a primitive root (mod p^2), with p odd, then g is primitive (mod p^h) for all $h \geq 3$).

Then we may proceed as in Example 7.3.

There are precisely $\varphi(p^h) = p^h - p^{h-1}$ characters (mod p^h), namely

$$\chi_k : (\mathbb{Z}/p^h\mathbb{Z})^\times \longrightarrow \mathbb{C}^\times$$

$$\chi_k(n) = e\left(\frac{k\,\mathrm{ind}_g(n)}{\varphi(p^h)}\right) \qquad (0 \leq k < \varphi(p^h))$$

$$X(p^h) := \langle \chi_1 \rangle \cong C_{\varphi(p^h)} \,\hat{=}\, (\mathbb{Z}/p^h\mathbb{Z})^\times .$$

3

Example 7.5. Let $q = 2^h$ with $h \geq 3$, and consider Dirichlet characters χ modulo 2^h.

In this case $(\mathbb{Z}/2^h\mathbb{Z})^\times$ is generated by the elements -1 and 5, and is isomorphic to

$$C_2 \times C_{2^{h-2}}. \qquad \left(5^{2^{h-2}} \equiv 1 \pmod{2^h}\right).$$

Given an integer n with $(n, 2^h) = 1$, there is a pair of integers $\underset{\operatorname{ind}^*_{-1} n}{\underset{\parallel}{\nu}}$ and $\underset{\operatorname{ind}^*_{5} n}{\underset{\parallel}{\mu}}$ having the property ~~that~~

$$n \equiv (-1)^\nu 5^\mu \pmod{2^h}.$$

We then have

$$\chi(n) = \chi((-1)^\nu 5^\mu) = \chi(-1)^\nu \chi(5)^\mu.$$

So the character is determined by its value at -1 & 5.

Moreover $\chi(-1)^2 = \chi(1) = 1$, $\chi(5)^{2^{h-2}} = \chi(1) = 1$.

④ It follows that there are precisely $2^{h-1} = \varphi(2^h)$ characters modulo 2^h, namely

$$\chi_{l,k} : (\mathbb{Z}/2^h\mathbb{Z})^\times \longrightarrow \mathbb{C}^\times$$

defined by taking

$$\chi_{l,k}(n) = e\left(\frac{l \operatorname{ind}^*_{-1}(n)}{2} + \frac{k \operatorname{ind}^*_5(n)}{2^{h-2}} \right) \qquad (l \in \{0,1\},\ 0 \le k < 2^{h-2}).$$

$$X(2^h) = \{ \chi_{l,k} : l \in \{0,1\},\ 0 \le k \le 2^{h-2}-1 \} = \langle \chi_{0,1}, \chi_{1,0} \rangle$$

$$\cong C_2 \times C_{2^{h-2}} \cong (\mathbb{Z}/2^h\mathbb{Z})^\times.$$

Example 7.6: Let $q \in \mathbb{N}$ with $q > 1$, and consider the character χ modulo q.

By the Chinese Remainder Theorem, one finds that

⑤

$$(\mathbb{Z}/q\mathbb{Z})^\times \cong \bigotimes_{p^\alpha \| q} (\mathbb{Z}/p^\alpha\mathbb{Z})^\times$$

so that $(\mathbb{Z}/q\mathbb{Z})^\times$ is generated by suitable representatives of primitive roots g_p, for $p^\alpha \| q$, with -1 and 5 if $8 | q$. It follows ~~that~~ there are precisely $\varphi(q)$ Dirichlet characters modulo q, namely

$$\chi_{\underline{k}} : (\mathbb{Z}/q\mathbb{Z})^\times \longrightarrow \mathbb{C}^\times \qquad (\text{when } 8 \nmid q)$$

defined by taking

$$\chi_{\underline{k}}(n) = e\left(\sum_{p^\alpha \| q} \frac{k_{p^\alpha} \, \mathrm{ind}_{g_p}(n)}{\varphi(p^\alpha)} \right)$$

with $0 \leq k_{p^\alpha} \leq \varphi(p^\alpha) - 1$.

and

$$\chi_{\underline{\ell}, \underline{k}} : (\mathbb{Z}/q\mathbb{Z})^\times \longrightarrow \mathbb{C}^\times \qquad (\text{when } 8 | q)$$

defined by taking

$$\chi_{\underline{l},\underline{k}}(n) = e\left(\frac{l_0 \operatorname{ind}^*_{-1}(n)}{2} + \frac{l_1 \operatorname{ind}^*_5(n)}{2^{h-2}} + \sum_{\substack{p^\alpha \| q \\ p \text{ odd}}} \frac{k_{p^\alpha} \operatorname{ind}_{g_p}(n)}{\varphi(p^\alpha)}\right),$$

with $2^h \| q$ with $h \geq 3$, wherein $l_0 \in \{0, 1\}$, $0 \leq l_1 < 2^{h-2}$, and $0 \leq k_{p^\alpha} \leq \varphi(p^\alpha) - 1$ (p odd).

Notation $X(q) := \{\chi_{\underline{l},\underline{k}}\} \cong (\mathbb{Z}/q\mathbb{Z})^\times$.

<u>Theorem 7.7.</u> The multiplicative group $(\mathbb{Z}/q\mathbb{Z})^\times$ of reduced residue classes modulo q has $\varphi(q)$ Dirichlet characters. If χ is such a character, then

(7)

$$\sum_{\substack{n=1\\(n,q)=1}}^{q} \chi(n) = \begin{cases} \phi(q), & \text{when } \chi = \chi_0, \\ 0, & \text{when } \chi \neq \chi_0. \end{cases}$$

Moreover, when $(n,q)=1$, one has

$$\sum_{\chi \in \underline{X}(q)} \chi(n) = \begin{cases} \phi(q), & \text{when } n \equiv 1 \pmod{q}, \\ 0, & \text{otherwise}. \end{cases}$$

<u>Proof.</u> When $\chi = \chi_0$, the first assertion is plain. Suppose then that $\chi \neq \chi_0$. There exists an integer m with $(m,q)=1$ and $\chi(m) \neq 1$. But then, by noting that multiplication by m permutes the reduced residues modulo q, we deduce ~~that~~

$$\sum_{\substack{n=1\\(n,q)=1}}^{q} \chi(n) = \sum_{\substack{n=1\\(n,q)=1}}^{q} \chi(mn) = \chi(m) \sum_{\substack{n=1\\(n,q)=1}}^{q} \chi(n).$$

whence

$$\underbrace{(\chi(m)-1)}_{\neq 0}\sum_{\substack{n=1\\(n,q)=1}}^{q}\chi(n)=0 \quad\Rightarrow\quad \sum_{\substack{n=1\\(n,q)=1}}^{q}\chi(n)=0.$$

□

For the second assertion, when $n \equiv 1 \pmod q$, one has

$$\sum_{\chi\in X(q)}\chi(n) = \sum_{\chi\in X(q)}\chi(1) = \mathrm{card}(X(q)) = \varphi(q).$$ ✓

Also, when $(n,q)=1$ and $n \not\equiv 1 \pmod q$, ~~there~~ is a character $\chi_1 \pmod q$ with $\chi_1(n) \neq 1$. (by our characterisation of Dirichlet characters). Thus

$$\sum_{\chi\in X(q)}\chi(n) = \sum_{\chi\in X(q)}(\chi_1\chi)(n) = \chi_1(n)\sum_{\chi\in X(q)}\chi(n)$$

use group structure of $X(q)$

(9)

$$\Rightarrow \quad \underbrace{(\chi_1(n)-1)}_{\neq 0} \sum_{\chi \in X(q)} \chi(n) = 0 \rightarrow \sum_{\chi \in X(q)} \chi(n) = 0. \quad \square$$

//

21 Sep 2020

① Note on problem B4 (ii) of Problems 2

(ii) Prove that there is a real number $c = c(k,r) > 0$ for which one has

$$\prod_{p \le x} (1 - k^r p^{-1}) \left(\sum_{h=0}^{\infty} \tau_k(p^h)^r p^{-h} \right) = c + O(1/x)$$

$$\downarrow$$

$$\prod_{p \le x} (1 - p^{-1})^{k^r} \left(\sum_{h=0}^{\infty} \tau_k(p^h)^r p^{-h} \right) = c + O(1/x).$$

Recall: Characters χ mod q (Dirichlet).

$8 \nmid q$: $\chi_{\underline{k}}(n) = e\left(\sum_{p^\alpha \| q} \frac{k_{p^\alpha} \operatorname{ind}_{g_p}(n)}{\varphi(p^\alpha)} \right)$ $(0 \le k_{p^\alpha} \le \varphi(p^\alpha) - 1)$

$8 \mid q$: $\chi_{\underline{l}, \underline{k}}(n) = e\left(\frac{l_0 \operatorname{ind}^*_{-1}(n)}{2} + \frac{l_1 \operatorname{ind}^*_5(n)}{2^{h-2}} + \sum_{\substack{p^\alpha \| q \\ p \text{ odd}}} \frac{k_{p^\alpha} \operatorname{ind}_{g_p}(n)}{\varphi(p^\alpha)} \right),$

when $2^h \| q$ with $h \ge 3$, with $l_0 \in \{0, 1\}, 0 \le l_1 < 2^{h-2}$; $0 \le k_{p^\alpha} < \varphi(p^\alpha)$ (p odd).

②

Theorem 7.7 $X(q) \cong (\mathbb{Z}/q\mathbb{Z})^{\times}$ (there are $\varphi(q)$ Dirichlet characters).

Also,

$$\sum_{\substack{n=1 \\ (n,q)=1}}^{q} \chi(n) = \begin{cases} \varphi(q), & \text{when } \chi = \chi_0, \\ 0, & \text{when } \chi \neq \chi_0, \end{cases}$$

$$\sum_{\chi \in X(q)} \chi(n). = \begin{cases} \varphi(q), & \text{when } n \equiv 1 \pmod{q}, \\ 0, & \text{when } n \not\equiv 1 \pmod{q} \text{ \& } (n,q)=1. \end{cases}$$

Theorem 7.8. (i) If χ_i is a character mod q_i $(i=1,2)$, and $q = [q_1, q_2]$, $\chi_3(n) = \chi_1(n)\chi_2(n)$ for $n \in \mathbb{Z}$, then $\chi_3 : \mathbb{Z} \to \mathbb{C}$ is a character mod q;

(ii) If $q = q_1 q_2$ with $(q_1, q_2) = 1$, and χ is a character modulo q, then there exist unique characters χ_i (mod q_i) $(i=1,2)$ such that

$\chi(n) = \chi_1(n)\chi_2(n)$ for all n.

③

<u>Exercise:</u> Analogue of Parseval / Plancharel:

(i) $$\frac{1}{\varphi(q)} \sum_{\chi \in X(q)} \left| \sum_{n=1}^{q} c_n \chi(n) \right|^2 = \sum_{\substack{n=1 \\ (n,q)=1}}^{q} |c_n|^2 \qquad (c_n \in \mathbb{C})$$

(ii) $$\frac{1}{\varphi(q)} \sum_{n=1}^{q} \left| \sum_{\chi \in X(q)} c_\chi \chi(n) \right|^2 = \sum_{\chi \in X(q)} |c_\chi|^2 \qquad (c_\chi \in \mathbb{C}).$$

---

§8. Dirichlet L-functions and Dirichlet's theorem on primes

<u>Dirichlet L-function:</u> $L(s,\chi)$ associated with χ modulo q is defined for $\sigma > 1$ by

$$L(s,\chi) = \sum_{n=1}^{\infty} \chi(n) n^{-s}. \qquad (8.1)$$

(Cf. $\zeta(s)$). When $\chi \neq \chi_0$, have

$$\sum_{1 \le n \le kq} \chi(n) = k \sum_{n=1}^{q} \chi(n) = 0,$$

④ whence

$$\Big| \sum_{1 \le n \le x} \chi(n) \Big| \le q$$

Thus, by Theorem 2.4 implies that (8.1) converges for $\sigma > 0$. (i.e. $\sigma_c = 0$).

Also, when $\chi = \chi_0$: $L(s, \chi_0)$ resembles $\zeta(s)$ closely, so $\sigma_c = 1$.

The absolute convergence of (8.1) for $\sigma > 1$ ensures that one has the Euler product

$$L(s,\chi) = \prod_p \sum_{h=0}^{\infty} \frac{\chi(p^h)}{p^{hs}} = \prod_p \sum_{h=0}^{\infty} \frac{\chi(p)^h}{p^{hs}}$$

$$= \prod_p \left(1 - \chi(p)p^{-s}\right)^{-1} \quad (\sigma > 1). \quad \text{—— (8.2)}$$

In particular,

$$L(s,\chi_0) = \sum_{\substack{n=1 \\ (n,q)=1}}^{\infty} n^{-s} = \zeta(s) \prod_{p \mid q} (1 - p^{-s}).$$

5 This relation shows ~~that~~ $L(s,\chi_0)$ is analytic for $\sigma > 0$ except for a simple pole at $s=1$.

<u>Theorem 8.1</u> If $\chi \neq \chi_0$, then $L(s,\chi)$ is analytic for $\sigma > 0$. Meanwhile, when $\sigma > 0$ the function $L(s,\chi_0)$ is analytic except for a simple pole at $s=1$ with residue $\varphi(q)/q$. In either case, when $\sigma > 1$, one has

$$\log L(s,\chi) = \sum_{n=2}^{\infty} \frac{\Lambda(n)}{\log n} \chi(n) n^{-s}$$

and

$$-\frac{L'}{L}(s,\chi) = \sum_{n=1}^{\infty} \Lambda(n)\chi(n) n^{-s}.$$

<u>Proof.</u> Analogous to treatment of $\log \zeta(s)$ & $-\frac{\zeta'}{\zeta}(s)$. The only claim to check concerns the residue of $L(s,\chi_0)$ at $s=1$. For this we have

⑥

$$\lim_{s\to 1}\Big((s-1)\underbrace{\zeta(s)\prod_{p|q}(1-p^{-s})}_{L(s,\chi)}\Big) = \prod_{p|q}(1-p^{-1}) = \frac{\varphi(q)}{q}. //$$

Interested in primes $p \equiv a \pmod q$ with $(a,q)=1$, hence in

$$\sum_{\substack{1\le n\le x\\ n\equiv a \pmod q}} \Lambda(n)$$ as an analogue of $\psi(x) = \sum_{1\le n\le x} \Lambda(n)$.

Observe that

$$\frac{1}{\varphi(q)}\sum_{\chi\in X(q)} \bar{\chi}(a)\chi(n) = \begin{cases} 1, & \text{when } n\equiv a \pmod q, \\ 0, & \text{when } n \not\equiv a \pmod q. \end{cases}$$

$$\frac{1}{\phi(q)}\sum_{\chi\in X(q)} \chi(a^{-1}n) = \begin{cases} 1, & \text{when } a^{-1}n\equiv 1 \pmod q \\ 0, & \text{when } a^{-1}n \not\equiv 1 \pmod q \end{cases}$$

Thus we have

⑥ ⓐ

$$\sum_{\substack{n=1 \\ n \equiv a \pmod q}}^{\infty} \Lambda(n) n^{-s} = \frac{1}{\varphi(q)} \sum_{n=1}^{\infty} \Lambda(n) n^{-s} \sum_{\chi \in X(q)} \bar{\chi}(a) \chi(n)$$

$$= \frac{1}{\varphi(q)} \sum_{\chi \in X(q)} \bar{\chi}(a) \left(- \frac{L'}{L}(s, \chi) \right) \quad (\sigma > 1).$$

$$\text{———— } (18 \cdot 4)$$

Strategy for proving that there are ∞ly many primes $p \equiv a \pmod q$. When $(q, a) = 1$.

Tempting to believe that when $\chi \neq \chi_0$,

$$- \frac{L'}{L}(s, \chi) = \sum_{n=1}^{\infty} \frac{\Lambda(n) \chi(n)}{n^s}$$ converges when $\sigma > 0$.

Indeed, if $\sigma > 1$ the series ~~is~~ analytic (since true for $L(s, \chi)$ and also $\log L(s, \chi)$ provided that $L(s, \chi) \neq 0$ in $\sigma > 0$.

⑦ Then provided that $L(1,\chi) \neq 0$, one has

$$\lim_{s \to 1+} \frac{L'}{L}(s,\chi) = \frac{L'}{L}(1,\chi) < \infty. \qquad (\chi \neq \chi_0).$$

Moreover, the function $L(s,\chi_0)$ has a simple pole at $s=1$, and hence $-\frac{L'}{L}(s,\chi_0)$ also has a simple pole at $s=1$ with residue 1. [$L(s,\chi_0) \sim \frac{\varphi(q)/q}{s-1} + O(1)$

$$\downarrow$$

$$\log L(s,\chi_0) \sim \log\left(\frac{1}{s-1}\right) + O(1)$$

$$\downarrow$$

$$-\frac{L'}{L}(s,\chi_0) = \frac{1}{s-1} + \cdots \ ]$$

Hence (8.4) yields

$$\sum_{\substack{n=1 \\ n \equiv a \ (\mathrm{mod}\ q)}}^{\infty} \Lambda(n) n^{-s} = \frac{1}{\varphi(q)} \sum_{\chi \in X(q)} \bar{\chi}(a) \left(-\frac{L'}{L}(s,\chi)\right)$$

⑧

$$= \frac{1}{\varphi(q)} \cdot (s-1)^{-1} + c_0 + \sum_{i \geq 1} c_i (s-1)^i,$$

as $s \to 1+$,

for some $c_i \in \mathbb{C}$.

This $\to +\infty$ as $s \to 1+$, hence $\displaystyle\sum_{\substack{n=1 \\ n \equiv a \pmod q}}^{\infty} \frac{\Lambda(n)}{n} = +\infty$.

Problem: Prove $L(1, \chi) \neq 0$ for $\chi \neq \chi_0$.

23 Sep 2020

①

Recall: Dirichlet L-function $L(s,\chi) = \sum_{n=1}^{\infty} \chi(n) n^{-s}$

$\chi = \chi_0$: $L(s,\chi_0) = \zeta(s) \prod_{p \mid q} (1-p^{-s}) = \frac{\varphi(q)/q}{s-1} + c_0 + c_1(s-1) + \dots$

$\chi \neq \chi_0$: analytic for $\sigma > 0$.

$$\sum_{\substack{n=1 \\ n \equiv a \pmod q}}^{\infty} \Lambda(n) n^{-s} = \frac{1}{\varphi(q)} \sum_{\chi \in X(q)} \bar{\chi}(a) \left(-\frac{L'}{L}(s,\chi) \right). \quad (\sigma > 1). \quad \text{—— (8.4)}$$

Strategy for proving infinitely many primes $p \equiv a \pmod q$ for $(a,q)=1$.

If $\chi \neq \chi_0$ and $L(1,\chi) \neq 0$, then $\operatorname{Log} L(s,\chi)$ analytic in nhd of $s=1$, whence $\frac{L'}{L}(s,\chi)$ analytic in same nhd. So

$$\lim_{s \to 1+} \frac{L'}{L}(s,\chi) = \frac{L'}{L}(1,\chi) < \infty.$$

Thus, by examining Laurent series expansions around $s=1$ get

$$\sum_{\substack{n=1 \\ n \equiv a \pmod q}}^{\infty} \Lambda(n) n^{-s} = \underbrace{\frac{1}{\varphi(q)} (s-1)^{-1}}_{\searrow +\infty.} + O(1) \quad \text{as } s \to 1+$$

②

__Theorem 8.2__ (Dirichlet) When χ is a Dirichlet character modulo q with $\chi \neq \chi_0$, one has $L(1,\chi) \neq 0$.

__Proof.__ Argument divides into two halves – we'll meet interesting lemmata in each half.

• A character is called __real__ if all of its values are real, and a character is called __complex__ if at least one of its values not real.

__Case 1: χ complex__

Note that if χ is complex with $L(1,\chi)=0$ (and proceed by contradiction) then $\overline{\chi}$ is another character with $\overline{\chi} \neq \chi$.
(if $\chi(n) \notin \mathbb{R} \Rightarrow \overline{\chi}(n) \neq \chi(n)$, so $\overline{\chi} \neq \chi$).

Then $\overline{\chi}$ is a character with $\overline{\chi} \neq \chi$ and $L(1,\overline{\chi})=0$, as we now show. One can proceed by brute force to see this: consider partial sums of $L(\sigma,\overline{\chi})$ as $\sigma \to 1+$

③ This shows that

$$\lim_{\sigma \to 1+} L(\sigma, \bar{\chi}) = \lim_{\sigma \to 1+} \overline{L(\sigma, \chi)} = \overline{\lim_{\sigma \to 1+} L(\sigma, \chi)} = 0.$$

whence $L(1, \bar{\chi}) = 0$.

[High-brow approach, can apply Schwarz reflection principle to show ~~that~~ $\overline{L(s,\chi)} = L(\bar{s}, \bar{\chi})$ — hint: $L(s,\chi) \pm L(s, \bar{\chi})$].

Observe ~~that~~ for $\sigma > 1$,

$$\log \Big(\prod_{\chi \in X(q)} L(s,\chi) \Big) = \sum_{\chi \in X(q)} \log L(s,\chi)$$

$$= \sum_{\chi \in X(q)} \sum_{n=2}^{\infty} \frac{\Lambda(n)}{\log n} \chi(n) n^{-s},$$

whence

$$\prod_{\chi \in X(q)} L(s,\chi) = \exp\Big(\varphi(q) \sum_{\substack{n=2 \\ n \equiv 1 \ (\mathrm{mod}\ q)}}^{\infty} \frac{\Lambda(n)}{\log n} n^{-s} \Big).$$

④

For $s=\sigma>1$, therefore, the sum here is a non-negative real number, whence

$$\prod_{\chi \in X(q)} L(\sigma,\chi) \geq 1. \qquad (8.5)$$

Consider the Laurent expansions of $L(s,\chi)$ about $s=1$ for $\chi \in X(q)$. The function $L(s,\chi_0)$ is analytic for $\sigma>0$ except for a simple pole at $s=1$ with residue $\varphi(q)/q$, so

$$L(s,\chi_0) = \frac{\varphi(q)/q}{s-1} + c_0 + c_1(s-1) + \cdots,$$

for suitable $c_i \in \mathbb{C}$. Meanwhile, when $\chi \neq \chi_0$, the function $L(s,\chi)$ is analytic for $\sigma>0$, whence

$$L(s,\chi) = c_0(\chi) + c_1(\chi)(s-1) + \cdots,$$

for suitable $c_i(\chi) \in \mathbb{C}$.

⑤ Then in (8.5) we have

$$\left(\frac{\varphi(q)/q}{s-1} + c_0 + \cdots \right) \prod_{\chi \neq \chi_0} \left(c_0(\chi) + c_1(\chi)(s-1) + \cdots \right) \geq 1$$

(for $\sigma > 1$).

But ~~then~~ $c_0(\chi)$ can be zero for at most one χ with $\chi \neq \chi_0$, for otherwise we would have l.h.s. divisible by $(s-1)$ (i.e. l.h.s. vanishes at $s=1$). This contradicts our hypothesis ~~that~~ $L(1,\chi)=0$ (and $L(1,\bar{\chi})=0$ as a consequence).

Hence $L(1,\chi) \neq 0$ for $\chi \neq \chi_0$. $\square$

<u>Case 2: χ real and $\chi \neq \chi_0$.</u>

- If χ is a real character, then $\chi^2 = \chi_0$. A character is called <u>quadratic</u> if it has order 2

⑥ in $X(q)$, where $\chi^2 = \chi_0$ but $\chi \neq \chi_0$.

In order to show that $L(1,\chi) \neq 0$, we must consider an auxiliary function. To motivate, consider that if $L(1,\chi)=0$,

$$L(s,\chi) = c_1(\chi)(s-1) + c_2(\chi)(s-1)^2 + \cdots,$$

and

$$\zeta(s) = \frac{1}{s-1} + C_0 + \cdots.$$

Thus

$$\zeta(s) L(s,\chi) = c_1(\chi) + b_1(s-1) + b_2(s-1)^2 + \cdots, \tag{8.6}$$

where $b_i \in \mathbb{C}$ are suitable constants.

One has

$$\zeta(s) L(s,\chi) = \sum_{n=1}^{\infty} r(n) n^{-s} \quad (\sigma > 1),$$

where

$$r(n) = \sum_{d|n} \chi(d).$$

(7) The function $r(n)$ is multiplicative, one has

$$r(p^h) = \begin{cases} 1, & \text{when } p \mid q \quad (\chi(p)=0 \text{ if } p\mid q) \\ h+1, & \text{when } \chi(p) = +1, \\ 1, & \text{when } \chi(p) = -1 \text{ and } 2\mid h, \\ 0, & \text{when } \chi(p) = -1 \text{ and } 2\nmid h. \end{cases}$$

[Note: $r(p^{2\ell}) = \sum_{m=0}^{2\ell} \underbrace{\chi(p)^m}_{(-1)^m} = (1-1)+\dots+(1-1)+1 = +1$],

Thus we see that $r(n) \geq 0$ for all $n \in \mathbb{N}$, and moreover $r(n^2) \geqslant 1$ for all $n \in \mathbb{N}$.

Let σ_c denote the abscissa of convergence of $\sum_{n=1}^{\infty} r(n) n^{-s}$. Then since

$$\sum_{n=1}^{\infty} r(n) n^{-1/2} \geq \sum_{m=1}^{\infty} r(m^2) m^{-1} \geq \sum_{m} m^{-1} = +\infty,$$

⑧. ~~Then~~ then $1 \geq \sigma_c \geq \frac{1}{2}$. However, when $L(1,\chi)=0$, the function $\zeta(s)L(s,\chi)$ is analytic for $\sigma>0$, including the point $s=1$. This is untenable — next time using Landau's Lemma.

25 Sep 2020

① • Problem Sheet 2 due — get that in via Brightspace soon

Recall: **Theorem 8.2.** (Dirichlet, 1837) (Proof here via Landau, 1906).
When χ is a character modulo q with $\chi \neq \chi_0$, one has $L(1,\chi) \neq 0$.

Proof. (so far). Two cases:

Case 1: χ complex: Have $\displaystyle\prod_{\chi \in X(q)} L(\sigma, \chi) \geq 1 \quad (\sigma > 1)$. —— (8.5)

Now
$$\begin{aligned} L(s,\chi_0) &= \frac{\varphi(q)/q}{s-1} + c_0 + c_1(s-1) + \dots \\ &= \zeta(s) \prod_{p \mid q} (1 - p^{-s}) \end{aligned}$$
has simple pole at $s=1$,

and $L(s,\chi)$ analytic in $\sigma > 0$ for $\chi \neq \chi_0$.

$\xrightarrow{(8.5)}$ at most one χ such that $L(1,\chi) = 0$.

But if χ is complex and $L(1,\chi) = 0$, then $L(1,\overline{\chi}) = 0$ & $\chi \neq \overline{\chi}$.

※.

Case 2: χ real and $\chi \neq \chi_0$: Then $\chi(n) = \pm 1$ for $(n,q) = 1$ and
$$L(s,\chi) = c_1(\chi)(s-1) + c_2(\chi)(s-1)^2 + \dots$$

(2) $$\zeta(s) = \frac{1}{s-1} + C_0 + \dots .$$

Then $\zeta(s) L(s, \chi) = c_1(\chi) + b_1(s-1) + \dots$ has no singularity at $s = 1$, nor indeed for $\sigma > 0$.

Observe that $$\zeta(s) L(s, \chi) = \sum_{n=1}^{\infty} r(n) n^{-s} \quad (\sigma > 1),$$

where $$r(n) = \sum_{d \mid n} \chi(d),$$

So
$$r(p^h) = \left.\begin{cases} 1, & \text{when } p \mid q, \\ h+1, & \text{when } \chi(p) = +1, \\ 1, & \text{when } \chi(p) = -1 \text{ and } 2 \mid h, \\ 0, & \text{when } \chi(p) = -1 \text{ and } 2 \nmid h \end{cases}\right\} \geq 0$$

and $r(n^2) \geq 1$ for all $n \in \mathbb{N}$.

Hence
$$\sum_{n=1}^{\infty} r(n) n^{-1/2} \geq \sum_{n=1}^{\infty} r(n^2) n^{-1} \geq \sum_{n=1}^{\infty} n^{-1} = +\infty,$$

so $\sum r(n) n^{-s}$ has abscissa of convergence σ_c with some $1 \geq \sigma_c \geq \frac{1}{2}$.

③

<u>So far:</u> $\zeta(s)L(s,\chi)\left(=\sum r(n)n^{-s} \text{ for } \sigma>\sigma_c\right)$ is analytic for $\sigma>0$.

Since $\frac{1}{2}\le\sigma_c\le 1$, this means that $\zeta(s)L(s,\chi)$ is analytic at $s=\sigma_c$. But this yields a contradiction from:

<u>Lemma 8·3.</u> (Landau) Let $\alpha(s)=\sum_{n=1}^{\infty}a_n n^{-s}$ be a Dirichlet series whose abscissa of convergence σ_c is finite. Then provided that $a_n\geqslant 0$ for all $n\in\mathbb{N}$, the point $s=\sigma_c$ is a singularity of the function $\alpha(s)$.

<u>Proof:</u> By replacing a_n by $a_n n^{-\sigma_c}$, we may suppose that the abscissa of convergence of $\alpha(s)$ is $\sigma_c=0$ (wlog).

If $\alpha(s)$ were not analytic at $s=0$, then it would possess a singularity there, and the proof would be complete.

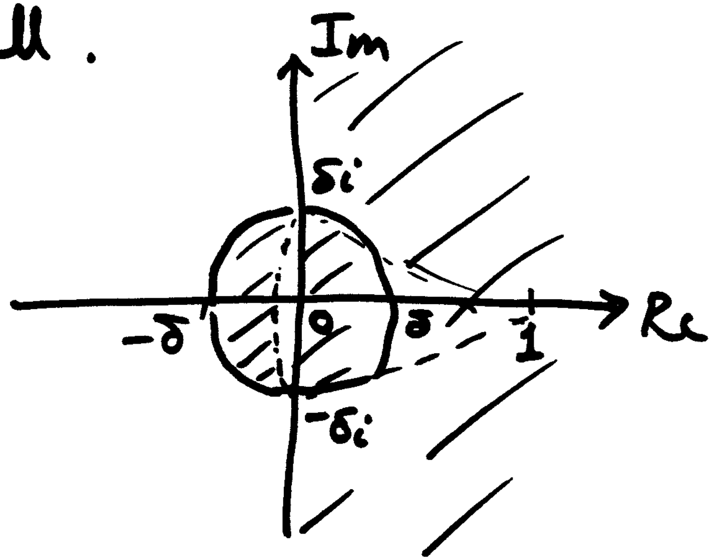
We may therefore suppose $\alpha(s)$ is analytic at $s=0$,

4

and hence also analytic in the domain

$$\mathcal{D} = \{ s \in \mathbb{C} : \sigma > 0 \text{ or } |s| < \delta \},$$

provided that δ is sufficiently small.



Our strategy is to show that

$\sum a_n n^{-s}$ converges to $\alpha(s)$

for some real number $\sigma < 0$.

This we achieve by considering a power series expansion for $\alpha(s)$ about $s = 1$.

Since $s=1$ is (well) inside the half-plane of convergence of $\alpha(s)$, we apply a Taylor expansion to obtain

$$\alpha(s) = \sum_{k=0}^{\infty} c_k (s-1)^k, \qquad (8.7)$$

where

$$c_k = \frac{\alpha^{(k)}(1)}{k!} = \frac{1}{k!} \sum_{n=1}^{\infty} a_n (-\log n)^k n^{-1}$$

(5) The radius of convergence of (8.7) is the distance from 1 to the nearest singularity of $\alpha(s)$.

Since $\alpha(s)$ is analytic in $\mathcal{D}$ (the shaded region), and the nearest points not in $\mathcal{D}$ are $\pm i\delta$, we see that the radius of convergence of (8.7) is $\geq \sqrt{1+\delta^2} = 1+\delta'$, say. Thus, when $|s-1| < 1+\delta'$, one has

$$\alpha(s) = \sum_{k=0}^{\infty} \frac{(1-s)^k}{k!} \sum_{n=1}^{\infty} a_n (\log n)^k n^{-1}.$$

If $s = \sigma < 1$, then all the terms are non-negative, and we may rearrange the summation for $-\delta' < \sigma < 1$ to deduce

$$\alpha(\sigma) = \sum_{n=1}^{\infty} a_n n^{-1} \sum_{k=0}^{\infty} \frac{(1-\sigma)^k (\log n)^k}{k!}$$

$$= \sum_{n=1}^{\infty} a_n n^{-1} \exp\left((1-\sigma)\log n\right) = \sum_{n=1}^{\infty} a_n n^{-\sigma}.$$

⑥ Then $\alpha(s)$ converges at $s = -\delta'/2$, contrary to our assumption ~~that~~ $\alpha(s)$ has abscissa of convergence $\sigma_c = 0$. Thus $\alpha(s)$ is <u>not</u> analytic at $s=0$, and instead has a singularity at $s=0$. //

$$L(1,\chi) \neq 0 \quad \text{for} \quad \chi \neq \chi_0. \quad \checkmark$$

<u>Corollary 8.4</u> (Dirichlet) If $a \in \mathbb{Z}$ and $q \in \mathbb{N}$ satisfy $(a,q)=1$, ~~then~~

$$\sum_{p \equiv a \pmod q} \frac{\log p}{p} = +\infty.$$

In particular, there are infinitely many primes p with $p \equiv a \pmod q$.

<u>Proof:</u> Recall from (8.4) ~~that~~ when $(a,q)=1$ and $\sigma > 1$, one has

⑦

$$\sum_{\substack{n=1 \\ n\equiv a \pmod q}}^{\infty} \Lambda(n)\, n^{-s} = -\frac{1}{\varphi(q)} \sum_{\chi\in X(q)} \bar{\chi}(a) \frac{L'}{L}(s,\chi)$$

↑ analytic for $\sigma>0$ except at zeros of $L(s,\chi)$, when $\chi\neq\chi_0$.

$$= \frac{1}{\varphi(q)(s-1)} + O_q(1) \quad \text{as } s\longrightarrow 1+$$

Then
$$\lim_{s\to 1+} \sum_{\substack{n=1 \\ n\equiv a \pmod q}}^{\infty} \Lambda(n)\, n^{-s} = +\infty,$$

and indeed
$$\sum_{\substack{n=1 \\ n\equiv a \pmod q}}^{\infty} \frac{\Lambda(n)}{n} = +\infty.$$

Moreover, the contribution from prime powers is

$$\sum_{\substack{p^k\equiv a \pmod q \\ k\geqslant 2}} \frac{\log p}{p^k} \leqslant \sum_p \log p \sum_{k=2}^{\infty} p^{-k} = \sum_p \frac{\log p}{p(p-1)} < \infty,$$

⑧

whence $$\sum_{p \equiv a \pmod q}^{\infty} \frac{\log p}{p} = +\infty .$$ //

Theorem 8.5. When $a \in \mathbb{Z}$ and $q \in \mathbb{N}$ satisfy $(a,q)=1$ and $x \geq 2$, one has

(a) $$\sum_{\substack{n \leq x \\ n \equiv a \ (\mathrm{mod}\ q)}} \frac{\Lambda(n)}{n} = \frac{1}{\varphi(q)} \log x + O_q(1) \quad \text{and} \quad \sum_{\substack{p \leq x \\ p \equiv a \ (\mathrm{mod}\ q)}} \frac{\log p}{p} = \frac{1}{\varphi(q)} \log x + O_q(1);$$

(b) $$\sum_{\substack{p \leq x \\ p \equiv a \ (\mathrm{mod}\ q)}} \frac{1}{p} = \frac{1}{\varphi(q)} \log\log x + b(q,a) + O_q\left(\frac{1}{\log x}\right),$$

where $$b(q,a) = \frac{1}{\varphi(q)} \Big(C_0 + \sum_{p \mid q} \log\left(1 - \frac{1}{p}\right) + \sum_{\chi \neq \chi_0} \bar{\chi}(a) \log L(1,\chi) \Big) - \sum_{\substack{p^k \equiv a \ (\mathrm{mod}\ q) \\ k > 1}} 1/kp^k \, ;$$

(c) $$\prod_{\substack{p \leq x \\ p \equiv a \ (\mathrm{mod}\ q)}} (1 - 1/p)^{-1} = C(q,a) (\log x)^{1/\varphi(q)} \left(1 + O_q\left(\frac{1}{\log x}\right)\right),$$

where

$$C(q,a) = \left(e^{C_0} \frac{\varphi(q)}{q} \prod_{\chi \neq \chi_0} \left(L(1,\chi)\right)^{\bar{\chi}(a)} \prod_p \left(1 - \frac{1}{p}\right)^{-\chi(p)} \left(1 - \frac{\chi(p)}{p}\right) \right)^{\frac{1}{\varphi(q)}}.$$

28 Sep 2020

1

Problems Class on Problem Sheet 1.

Theorem 2·4. Let $\alpha(s) = \sum_{n=1}^{\infty} a_n n^{-s}$ and write $A(x) = \sum_{n \leq x} a_n$.

(i) When $\sigma_c \geqslant 0$, one has

$$\sigma_c = \limsup_{x \to \infty} \frac{\log |A(x)|}{\log x} \, ;$$

(ii) When $\sigma_c < 0$, the function $A(x)$ is bounded;

(iii) When $\sigma > \max\{\sigma_c, 0\}$, one has

$$\sum_{n=1}^{\infty} a_n n^{-s} = s \int_1^{\infty} \frac{A(x)}{x^{s+1}} \, dx .$$

②

Q1 (i) Put $A(x) = \sum_{n \leq x} (\log n)^{2020} = \underset{\sim}{\underline{x(\log x)^{2020} + O(x(\log x)^{2019})}}$

To see this, ~~either~~ compare with integral, or proceed by R-S integration:

$$A(x) = \int_{2^-}^{x} (\log u)^{2020} \, d\Big(\sum_{n \leq u} 1\Big) = \int_{2^-}^{x} (\log u)^{2020} \, d\lfloor u \rfloor$$

$$= \lfloor u \rfloor (\log u)^{2020} \Big|_2^x - \int_2^x \lfloor u \rfloor \cdot 2020 (\log u)^{2019} / u \, du$$

$$= \underset{\sim}{\underline{x(\log x)^{2020} + O\Big(\int_2^x (\log u)^{2019} \, du\Big)}} . \square$$

$\boxed{\textbf{Easy: } x \ll A(x) \ll x^{1+\varepsilon} \text{ for any } \varepsilon > 0}$

Then since $A(x)$ is not bounded,

$$\sigma_c = \limsup_{x \to \infty} \frac{\log\big(x(\log x)^{2020} + O(x(\log x)^{2019})\big)}{\log x}$$

$$= \limsup_{x \to \infty} \frac{\log x + O(\log\log x)}{\log x} = \lim_{x \to \infty} 1 + O\Big(\frac{\log\log x}{\log x}\Big)$$

$$= 1 . \square$$

③ (ii) Put $A(x) = \sum_{n \le x} (n^2+1)^{3/2} = \sum_{n \le x} \left(n^3 + O(n)\right) = \frac{1}{4}x^4 + O(x^3)$.

Then since $A(x)$ is unbounded,

$$\sigma_c = \limsup_{x \to \infty} \frac{\log\left(\frac{1}{4}x^4 + O(x^3)\right)}{\log x} = \lim_{x \to \infty} \frac{4\log x + O(1)}{\log x}$$

$$= 4. \ \square$$

(iii) Put $A(x) = \sum_{n \le x} 2^{(\log n)^{3/2}}$. Then $2^{(\log(x-1))^{3/2}} \le A(x) \le x\, 2^{(\log x)^{3/2}}$.

Then since $A(x)$ is unbounded, we have

$$\sigma_c \ge \limsup_{x \to \infty} \frac{\log\left(2^{(\log(x-1))^{3/2}}\right)}{\log x} = \lim_{x \to \infty} \frac{(\log(x-1))^{3/2}\log 2}{\log x}$$

$$= (\log 2) \lim_{x \to \infty} \frac{(\log x)^{3/2}}{\log x} = +\infty. \ \square$$

④

Q2 (i) Put
$$A(x) = \sum_{n \le x} (-1)^{n-1} = \begin{cases} 1, & \text{when } \lfloor x \rfloor \text{ odd}, \\ 0. & \text{when } \lfloor x \rfloor \text{ even}. \end{cases}$$

Since $A(x)$ is bounded, one has $\sigma_c \le 0$. Also, since
$$A^+(x) := \sum_{n \le x} |(-1)^{n-1}| = \lfloor x \rfloor$$
is unbounded, one sees that
$$\sigma_a = \limsup_{x \to \infty} \frac{\log |A^+(x)|}{\log x} = \lim_{x \to \infty} \frac{\log \lfloor x \rfloor}{\log x} = 1.$$
Then since $\sigma_a \le \sigma_c + 1$, it follows that $\sigma_c = 0$. □

(ii) Put
$$A(x) = \sum_{n=1}^{x} \sin\left(\frac{n\pi}{2} + \frac{2\pi}{n}\right) = \begin{cases} 1 \text{ or } 0 \end{cases} + O\left(\sum_{n \le x} \frac{1}{n}\right)$$
$$\Vert$$
$$\begin{cases} 1 + O\left(\frac{1}{n}\right), & \text{when } n \equiv 1 \pmod 4 \\ O\left(\frac{1}{n}\right), & \text{when } 2 \mid n \\ -1 + O\left(\frac{1}{n}\right), & \text{when } n \equiv 3 \pmod 4 \end{cases}$$

⑤

Then, if $A(x)$ were unbounded, one obtains

$$\sigma_c = \limsup_{x \to \infty} \frac{\log |A(x)|}{\log x} \leq \lim_{x \to \infty} \frac{\log(C \log x)}{\log x} = 0.$$

If $A(x)$ were bounded, meanwhile, then $\sigma_c \leq 0$. Meanwhile, since

$$A^+(x) := \sum_{n \leq x} \left| \sin\left(\frac{n\pi}{2} + \frac{2\pi}{n}\right) \right| = \left\lfloor \frac{x}{2} \right\rfloor + O\left(\sum_{n \leq x} \frac{1}{n}\right),$$

we see ~~that~~

$$\sigma_a = \limsup_{x \to \infty} \frac{\log A^+(x)}{\log x} = \lim_{x \to \infty} \frac{\log(x/2 + O(\log x))}{\log x}$$

$$= 1.$$

Thus, since $\sigma_a \leq \sigma_c + 1$, it follows ~~that~~ $\sigma_c = 0$. □

(6)

Q3 (i) One has

$$\Big| \sum_{n>x} a_n n^{-s} \Big| \leq \sum_{n>x} |a_n| n^{-\sigma},$$

so if rhs $\to 0$ as $n \to \infty$ (as happens for $\sigma > \sigma_a$), then the lhs $\to 0$ as $x \to \infty$, so $\sum_n a_n n^{-s}$ converges. Thus $\sigma_c \leq \sigma_a$.

(ii) Suppose $\sigma > \sigma_c$, say $\sigma = \sigma_c + 2\varepsilon$. Then

$$\sum_n a_n n^{-\sigma_c - \varepsilon} < \infty \quad \Rightarrow \quad a_n n^{-\sigma_c - \varepsilon} \longrightarrow 0 \text{ as } n \longrightarrow \infty.$$

Thus
$$\sum_{n>x} |a_n| n^{-1-\sigma} < \sum_{n>x} n^{-1-(\sigma - \sigma_c - \varepsilon)} \quad \text{for } x \text{ suff. large}$$
$$= \sum_{n>x} n^{-1-\varepsilon} < \infty.$$

Then for any $\varepsilon > 0$, have $\sigma_a \leq \sigma_c + 1 + 2\varepsilon$, whence $\sigma_a \leq \sigma_c + 1$.

Q3

$$\prod_p \left(1 + \frac{\zeta(s)-1}{p^s}\right)$$

If $(m,n)=1$, ~~then~~

$$s_0(mn) = \prod_{p|mn} p = \left(\prod_{p|m} p\right)\left(\prod_{p'|n} p'\right)$$

distinct primes since $(m,n)=1$

$$= S_0(m)S_0(n). \rightarrow S_0(n) \text{ is mult.}$$

If $(m,n)=1$, ~~then~~

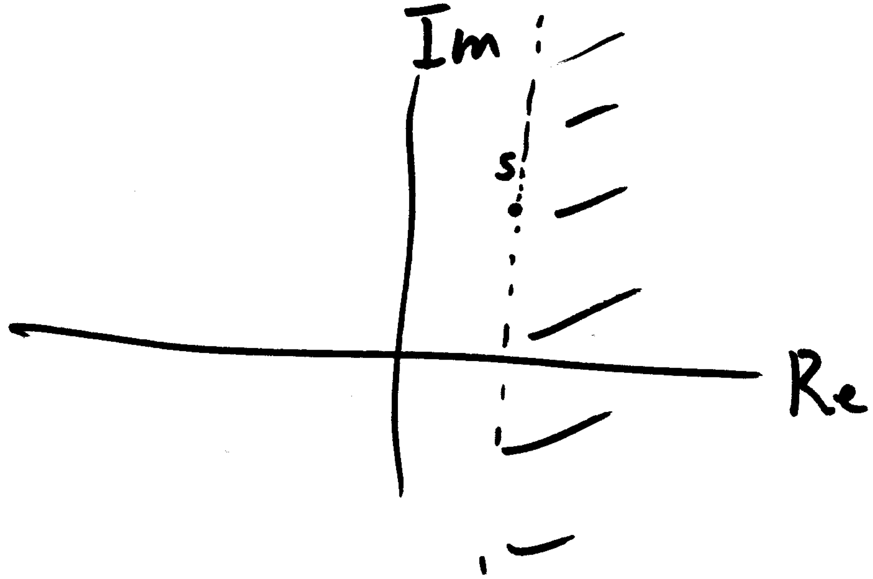
$$\tau(mn) = \sum_{d|mn} 1 = \sum_{d_1|m}\sum_{d_2|n} 1 = \left(\sum_{d_1|m} 1\right)\left(\sum_{d_2|n} 1\right) = \tau(m)\tau(n)$$

$(m,n)=1 \rightarrow d_1|m$ & $d_2|n$ satisfy $(d_1,d_2)=1$.

(i) $\sum_{n=1}^{\infty} (\log n)^{2020} n^{-s}$ (ii) $\sum_{n=1}^{\infty} (n^2+1)^{3/2} n^{-s}$.

(iii) $\sum_{n=1}^{\infty} 2^{(\log n)^{3/2}} n^{-s}$.

Q2 $\sum_{n=1}^{\infty} (-1)^{n-1} n^{-s}$ $\sum_{n=1}^{\infty} |(-1)^{n-1}||n^{-s}| = \sum_{n=1}^{\infty} n^{-\sigma}$



Recall: Theorem 8.2. (Dirichlet, 1837)
When χ is a character modulo q with $\chi \neq \chi_0$, one has $L(1, \chi) \neq 0$.

Corollary 8.4 (Dirichlet) When $a \in \mathbb{Z}$ and $q \in \mathbb{N}$ satisfy $(a,q)=1$, then
$$\sum_{p \equiv a \pmod q} \frac{\log p}{p} = +\infty.$$

Theorem 8.5. When $a \in \mathbb{Z}$ and $q \in \mathbb{N}$ satisfy $(a,q)=1$, and $x \geq 2$, one has

(a) $$\sum_{\substack{n \leq x \\ n \equiv a \pmod q}} \frac{\Lambda(n)}{n} = \frac{1}{\varphi(q)} \log x + O_q(1) \quad \text{and} \quad \sum_{\substack{p \leq x \\ p \equiv a \pmod q}} \frac{\log p}{p} = \frac{1}{\varphi(q)} \log x + O_q(1);$$

(b) $$\sum_{\substack{p \leq x \\ p \equiv a \pmod q}} \frac{1}{p} = \frac{1}{\varphi(q)} \log \log x + b(q,a) + O_q\left(\frac{1}{\log x}\right);$$

where $$b(q,a) = \frac{1}{\varphi(q)} \left(C_0 + \sum_{p \mid q} \log\left(1 - \frac{1}{p}\right) + \sum_{\chi \neq \chi_0} \bar{\chi}(a) \log L(1,\chi) \right) - \sum_{p^k \equiv a \pmod q \ (k>1)} 1/(k p^k) \quad;$$

②

(c) $$\prod_{\substack{p \le x \\ p \equiv a \pmod q}} (1 - 1/p)^{-1} = C(q,a)(\log x)^{1/\varphi(q)}\left(1 + O_q(1/\log x)\right),$$

where $$C(q,a) = \left(e^{c_0}\frac{\varphi(q)}{q}\prod_{\chi \ne \chi_0}\left(L(1,\chi)^{\bar{\chi}(a)}\prod_p\left(\left(1-\frac{1}{p}\right)^{-\chi(p)}\left(1-\frac{\chi(p)}{p}\right)\right)\right)\right)^{\frac{1}{\varphi(q)}}.$$

Proof: Exercise using Lemma 8.6. //

③

Lemma 8·6. Suppose that χ is a non-principal Dirichlet character. Then for $x \geq 2$, one has

(a) $$\sum_{n\leq x} \frac{\chi(n)\Lambda(n)}{n} \ll_\chi 1 \quad \text{and} \quad \sum_{p\leq x} \frac{\chi(p)\log p}{p} \ll_\chi 1;$$

(b) $$\sum_{p\leq x} \frac{\chi(p)}{p} = b(\chi) + O_\chi\left(\frac{1}{\log x}\right);$$

where $b(\chi) = \log L(1,\chi) - \sum_{p^k (k>1)} \frac{\chi(p)^k}{kp^k}$;

(c) $$\prod_{p\leq x} \left(1 - \frac{\chi(p)}{p}\right)^{-1} = L(1,\chi) + O_\chi\left(\frac{1}{\log x}\right).$$

The missing character is χ_0, and $\chi_0(n) = \begin{cases} 1, & (q,n)=1 \\ 0, & \text{else} \end{cases}$.

The corresponding expressions to those above can be calculated in terms of what we have already deduced for all primes. For example, observe that

④

$$\sum_{n \le x} \frac{\chi_0(n)\Lambda(n)}{n} = \sum_{n \le x} \frac{\Lambda(n)}{n} - \sum_{\substack{p^k \le x \\ p \mid q}} \frac{\log p}{p^k}$$

$$= \sum_{n \le x} \frac{\Lambda(n)}{n} + O_q(1)$$

$$= \log x + O_q(1).$$

Now, to get to ~~estimate~~ $\displaystyle\sum_{\substack{n \le x \\ n \equiv a \ (\mathrm{mod}\ q)}} \frac{\Lambda(n)}{n}$, consider

$$\frac{1}{\varphi(q)} \sum_{\chi \in X(q)} \bar{\chi}(a) \underbrace{\sum_{\substack{n \le x \\ n \equiv a \ (\mathrm{mod}\ q)}} \frac{\Lambda(n)\chi(n)}{n}}_{} \cancel{\quad} \longleftarrow \quad = \frac{1}{\varphi(q)} \log x + O_q(1).$$

either $\begin{cases} = \log x + O_q(1) & \text{when } \chi = \chi_0 \\ = O(1) & \text{when } \chi \ne \chi_0. \end{cases}$

4a

We look at the proof of part (a) of Lemma 8·6:

One has, for non-principal characters $\chi \neq \chi_0$,

$$\sum_{n \le x}' \frac{\chi(n)\log n}{n} = \sum_{n \le x} \frac{\chi(n)}{n} \sum_{d \mid n} \Lambda(d) \qquad n = md$$

$$= \sum_{d \le x} \frac{\Lambda(d)\,\chi(d)}{d} \sum_{m \le x/d}' \frac{\chi(m)}{m}.$$

The last sum is fairly easily estimated:

$$\sum_{m \le y} \frac{\chi(m)}{m} = L(1,\chi) - \sum_{m > y} \frac{\chi(m)}{m}.$$

To estimate the tail, put $S(x) = \sum_{n \le x} \chi(n)$. Then

$$\sum_{m > y} \frac{\chi(m)}{m} = \int_y^\infty u^{-1}\, d\,S(u) = -\frac{S(y)}{y} + \int_y^\infty S(u)\,u^{-2}\,du.$$

Since $|S(u)| \le q = O_q(1)$, we obtain this expression is $O_q\left(\frac{1}{y}\right)$.

Hence

⑤

$$\sum_{n\le x}\frac{\chi(n)\log n}{n} = \sum_{d\le x}\frac{\Lambda(d)\chi(d)}{d}\Big(\underset{\substack{\uparrow\\ \neq 0.}}{L(1,\chi)} + O_\chi\Big(\frac{1}{x/d}\Big)\Big)$$

$$= L(1,\chi)\sum_{d\le x}\frac{\Lambda(d)\chi(d)}{d} + \underbrace{\frac{1}{x}\,O\Big(\underbrace{\sum_{d\le x}\Lambda(d)}_{=O(x)}\Big)}_{=O(1)}$$

Thus

$$\sum_{d\le x}\frac{\Lambda(d)\chi(d)}{d} = \frac{1}{L(1,\chi)}\sum_{n\le x}\frac{\chi(n)\log n}{n} + O_\chi(1).$$

$$= \frac{1}{L(1,\chi)}\Big(-L'(1,\chi) - \sum_{n>x}\frac{\chi(n)\log n}{n}\Big) + O_\chi(1)$$

Again using R-S integration, we have

$$\sum_{n>x}\frac{\chi(n)\log n}{n} = \int_x^\infty \frac{\log u}{u}\, d\,S(u) \qquad S(u) = \sum_{n\le u}\chi(n)$$

$$= -\frac{S(x)\log x}{x} - \int_x^\infty S(u)\frac{(1-\log u)}{u^2}\,du$$

$$\ll_q \frac{\log x}{x}.$$

Then $\displaystyle\sum_{d \leq x} \frac{\Lambda(d)\chi(d)}{d} = -\frac{L'}{L}(1,\chi) + O_\chi(1) \ll_\chi 1.$ □

The remaining conclusions follow as in the argument of proof of Theorem 5.3. //

§9. Dirichlet series and Mellin transforms.

Consider a Dirichlet series $\alpha(s) = \sum_{n=1}^{\infty} a_n n^{-s}$ with abscissa of convergence σ_c, and consider partial sums

$$A(x) = \sum_{1 \leq n \leq x} a_n.$$

We saw (Theorem 2.4) that whenever $\sigma > \max\{0, \sigma_c\}$, one has

$$\alpha(s) = s\int_1^\infty A(x)\, x^{-s-1}\, dx. \qquad (9.1)$$

7

Definition 9.1 Suppose that $f: \mathbb{R} \to \mathbb{C}$. Then the **Mellin transform** $F(s)$ of the function $f(s)$ is defined by

$$F(s) = \int_0^\infty f(x)\, x^{s-1}\, dx .$$

Question: What about inverse Mellin transforms?

Definition 9.2. When $F: \mathbb{C} \to \mathbb{C}$, we define the **inverse Mellin transform** $f(x)$ of the function $F(s)$ by putting

$$f(x) = \frac{1}{2\pi i} \int_{\sigma_0 - i\infty}^{\sigma_0 + i\infty} F(s)\, x^{-s}\, ds$$

(when σ_0 is suitably large).

⑧

Perron's formula:

$$\alpha(s) = \sum_{n=1}^{\infty} a_n n^{-s}$$

$$\underset{\substack{\| \\ f(x)}}{\sum_{n \le x}{}' a_n} = \lim_{T \to \infty} \frac{1}{2\pi i} \int_{\sigma_0 - iT}^{\sigma_0 + iT} \alpha(s) \frac{x^s}{s}\, ds$$

when $\sigma_0 > \max\{0, \sigma_c\}$.

02 Oct 2020

①

Recall: §9. Dirichlet series and Mellin transforms.

Mellin transform: $f : \mathbb{R} \to \mathbb{C} \longmapsto F(s) = \int_0^\infty f(x) x^{s-1}\,dx$

Inverse Mellin transform: $F : \mathbb{C} \to \mathbb{C} \longmapsto f(x) = \frac{1}{2\pi i}\int_{\sigma_0 - i\infty}^{\sigma_0 + i\infty} F(s) x^{-s}\,ds.$

When $A(x) = \sum_{1 \le n \le x} a_n$ and $\alpha(s) = \sum_{n=1}^\infty a_n n^{-s}$ have special case

for $\sigma > \max\{0, \sigma_c\}$: $\alpha(s) = s\int_1^\infty A(x) X^{-s-1}\,dx.$ —— (9.1)

Theorem 9.1 (Perron's formula) Let $\alpha(s) = \sum_{n=1}^\infty a_n n^{-s}$ be a Dirichlet series with abscissa of convergence σ_c. Then whenever $\sigma_0 > \max\{0, \sigma_c\}$ and $x > 0$, one has

$$\sum_{n \le x}{}' a_n = \lim_{T \to \infty} \frac{1}{2\pi i} \int_{\sigma_0 - iT}^{\sigma_0 + iT} \alpha(s) \frac{x^s}{s}\,ds.$$

$$\parallel$$

$$\begin{cases} a_1 + \cdots + a_{[x]}, & \text{when } x \notin \mathbb{N} \\ a_1 + \cdots + a_{x-1} + \frac{1}{2} a_x, & \text{when } x \in \mathbb{N} \end{cases}$$

(1a) Application (later):

Take $\alpha(s) = -\frac{\zeta'}{\zeta}(s) = \sum_{n=1}^{\infty} \Lambda(n) n^{-s}$

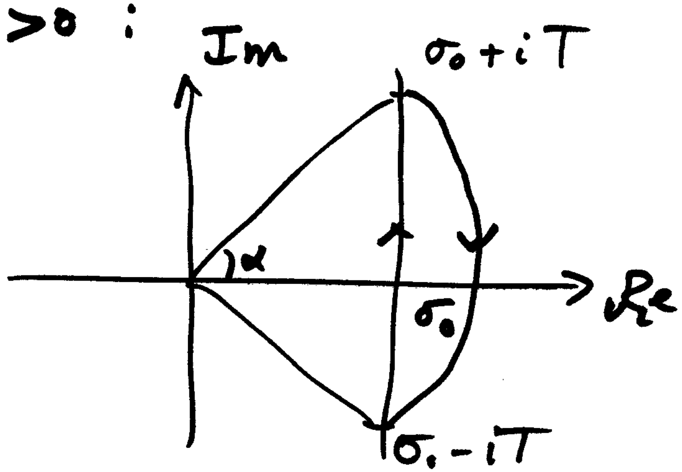
$\updownarrow$

$$\text{"}\psi(x)\text{"} = {\sum_{n\le x}}' \Lambda(n) = \lim_{T\to\infty} \frac{-1}{2\pi i}\int_{\sigma_0 - iT}^{\sigma_0+iT} \frac{\zeta'}{\zeta}(s)\,\frac{x^s}{s}\,ds \quad (\sigma_0>1).$$

$\uparrow$

can relate this to $\psi(x)$, hence to $\pi(x)$.

Also, when $y=1$ and $\sigma_0>0$:



$$\lim_{T\to\infty}\frac{1}{2\pi i}\int_{\sigma_0-iT}^{\sigma_0+iT}\frac{ds}{s} = \frac{1}{2}$$

$$\text{"} \quad \frac{1}{2\pi i}\int_{-\frac{1}{4}+O(\frac{1}{R})}^{1/4+O(\frac{1}{R})}\frac{d(Re(\theta))}{Re(\theta)} \longrightarrow \frac{1}{2} \text{ as } R\to\infty.$$

② <u>Proof</u>: The key observation is that, by the calculus of residues, one has

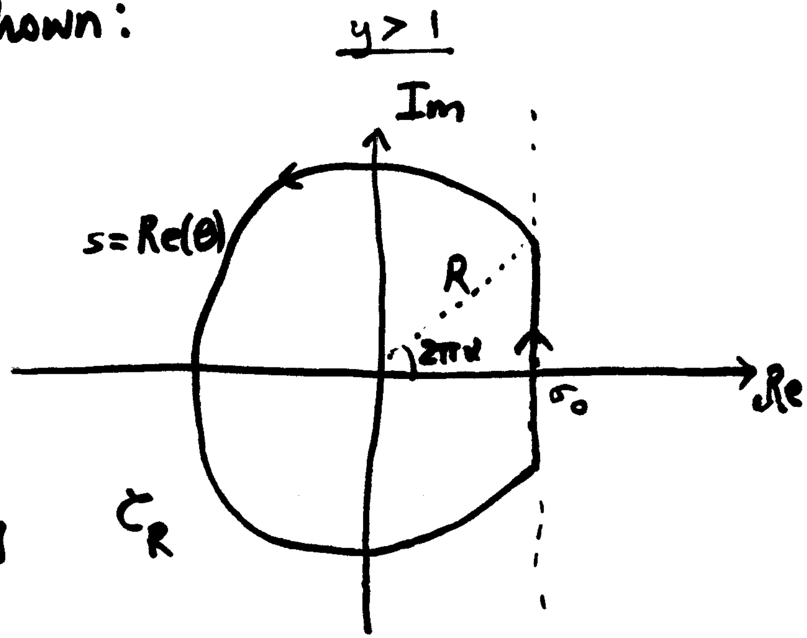
$$\frac{1}{2\pi i}\int_{\sigma_0 - i\infty}^{\sigma_0 + i\infty} y^s \frac{ds}{s} = \begin{cases} 1, & \text{when } y>1, \\ 0, & \text{when } 0<y<1, \end{cases}$$ provided that $\sigma_0>0$.

When $y>1$, consider contour $\mathcal{C}_R$ as shown:

Define real number α as shown by

$$R\cos(2\pi\alpha) = \sigma_0$$

$$\overset{\shortparallel}{R\sin\left(\frac{\pi}{2} - 2\pi\alpha\right)} = \sigma_0 \longrightarrow \alpha = \frac{1}{4} + O\left(\frac{1}{R}\right)$$

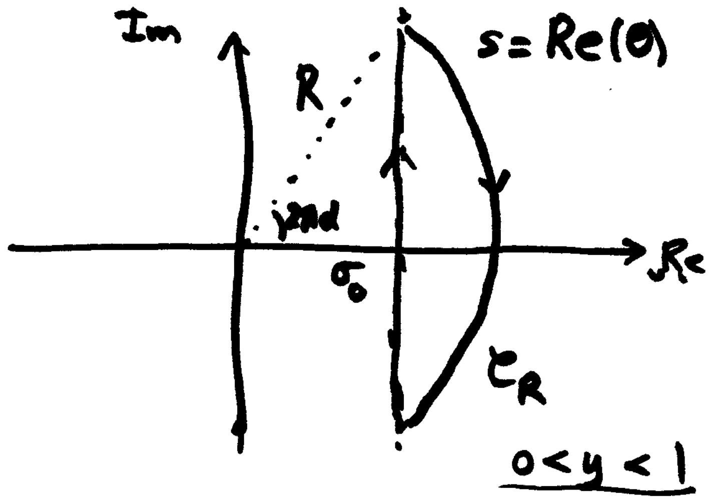


The circular arc within $\mathcal{C}_R$ contributes

$$\frac{1}{2\pi i}\int_{\alpha}^{1-\alpha} \frac{y^{Re(\theta)}}{Re(\theta)}\, d(Re(\theta))$$

$$\shortparallel$$

$$\int_{\alpha}^{1-\alpha} y^{Re(\theta)}\, d\theta$$



③

But

$$\left| \int_{\alpha}^{1-\alpha} y^{Re(\theta)} \, d\theta \right| \leq \int_{|\alpha - \frac{1}{4}| \leq C\sqrt{\log R}/R} y^{\sigma_0} \, d\theta + \int_{\substack{\alpha \\ |\alpha - \frac{1}{4}| > C\sqrt{\log R}/R}}^{1-\alpha} y^{-\sqrt{\log R}} \, d\theta$$

$\alpha = \frac{1}{4} + O\left(\frac{1}{R}\right)$

for a suitable $c > 0$.

Thus,

$$\left| \int_{\alpha}^{1-\alpha} y^{Re(\theta)} \, d\theta \right| \ll_{y,\sigma_0} \frac{\sqrt{\log R}}{R} \longrightarrow 0 \text{ as } R \to \infty .$$

But the function $\frac{y^s}{s}$ is analytic within the contour C_R except for a simple at $s = 0$, whence

$$\frac{1}{2\pi i} \int_{\sigma_0 - i\infty}^{\sigma_0 + i\infty} \frac{y^s}{s} \, ds = \lim_{R \to \infty} \frac{1}{2\pi i} \int_{C_R} \frac{y^s}{s} \, ds = \lim_{s \to 0} y^s = 1$$

③ⓐ Meanwhile, when $0 < y < 1$, a similar argument shows that

$$-\frac{1}{2\pi i}\int_{-\alpha}^{\alpha} \frac{y^{\mathrm{Re}(\theta)}}{\mathrm{Re}(\theta)}\, d(\mathrm{Re}(\theta)) = -\int_{-\alpha}^{\alpha} y^{\mathrm{Re}(\theta)}\, d\theta$$

$$\ll \int_{|\alpha - \frac{1}{4}| \leq C\sqrt{\log R}/R} y^{\sigma_0}\, d\theta + \int_{\substack{-\alpha \\ |\alpha - \frac{1}{4}| > C\sqrt{\log R}/R}}^{\alpha} y^{\sqrt{\log R}}\, d\theta$$

$$\ll_{y,\sigma_0} \sqrt{\log R}/R \to 0 \quad \text{as } R \to \infty.$$

In this instance, the function y^s/s is analytic within the contour $\mathcal{C}_R$ (no poles!), whence

$$\frac{1}{2\pi i}\int_{\sigma_0 - i\infty}^{\sigma_0 + i\infty} y^s \frac{ds}{s} = \lim_{R\to\infty} \frac{1}{2\pi i}\int_{\mathcal{C}_R} y^s \frac{ds}{s} = 0$$

④ Next return to proof of Perron's formula. Idea — truncate the series $\alpha(s)$. Thus, take N larger than $2x+2$ and $\sigma > \max\{0, \sigma_c\}$, we have

$$\alpha(s) = \underbrace{\sum_{n \le N} a_n n^{-s}}_{\alpha_1(s)} + \underbrace{\sum_{n > N} a_n n^{-s}}_{\alpha_2(s)}.$$

But then, when $\sigma_0 > \max\{0, \sigma_c\}$, we find ~~that~~

$$\sum_{n \le x}{}' a_n = \sum_{n \le x} a_n \lim_{T \to \infty} \int_{\sigma_0 - iT}^{\sigma_0 + iT} (x/n)^s \frac{ds}{s}$$

$$= \lim_{T \to \infty} \int_{\sigma_0 - iT}^{\sigma_0 + iT} \alpha_1(s) \frac{x^s}{s} ds. \qquad \square$$

We must deal with $\alpha_2(s)$. Apply Riemann-Stieltjes integration; for $\sigma > \max\{0, \sigma_c\}$, have

5

$$\alpha_2(s) = \int_{N+}^{\infty} u^{-s}\, d\,(A(u) - A(N))$$

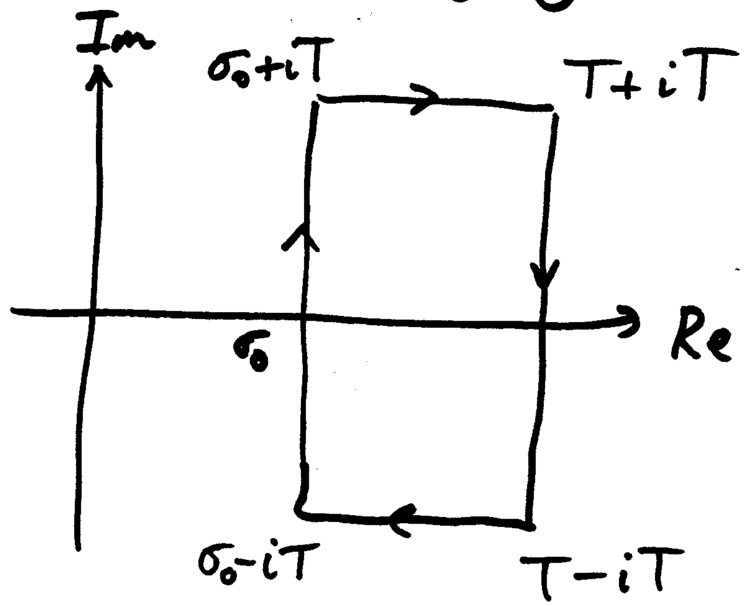
$$= s \int_{N}^{\infty} (A(u) - A(N))\, u^{-s-1}\, du.$$

Whenever $\theta > \max\{0, \sigma_c\}$, have $A(u) - A(N) \ll u^{\theta}$,

whence for $\sigma > \theta$, have

$$\alpha_2(s) \ll |s| \int_{N}^{\infty} u^{\theta-\sigma-1}\, du \ll \frac{|s|}{\sigma-\theta} N^{\theta-\sigma}.$$

$$\int_{\sigma_0 - iT}^{\sigma_0 + iT} \alpha_2(s) \frac{X^s}{s}\, ds$$



5 Oct 2020 | Recall: Theorem 9.1 (Perron's formula).

①

Let $\alpha(s) = \sum_{n=1}^{\infty} a_n n^{-s}$ be a Dirichlet series with abscissa of convergence σ_c. Then whenever $\sigma_0 > \max\{0, \sigma_c\}$ and $x > 0$, one has

$$\sum_{n\leq x}{}' a_n = \lim_{T\to\infty} \frac{1}{2\pi i} \int_{\sigma_0 - iT}^{\sigma_0 + iT} \alpha(s) \frac{x^s}{s}\, ds$$

Proof: We established that $\overbrace{\frac{1}{2\pi i} \int_{\sigma_0 - i\infty}^{\sigma_0 + i\infty}}$ $y^s \frac{ds}{s} = \begin{cases} 1, & \text{when } y > 1, \\ 0, & \text{when } 0<y<1, \\ \frac{1}{2}, & \text{when } y = 1. \end{cases}$

*Cauchy principal value.

Now — take $N > 2x + 2$ and $\sigma > \max\{0, \sigma_c\}$.

(1) Have $\alpha(s) = \sum_{n\leq N} a_n n^{-s} + \sum_{n>N} a_n n^{-s} = \alpha_1(s) + \alpha_2(s)$,

and

$$\sum_{n\leq x} a_n' = \sum_{n\leq x} a_n \lim_{T\to\infty} \int_{\sigma_0 - iT}^{\sigma_0 + iT} \left(\frac{x}{n}\right)^s \frac{ds}{s} = \lim_{T\to\infty} \int_{\sigma_0 - iT}^{\sigma_0 + iT} \alpha_1(s) \frac{x^s}{s}\, ds.$$

(2) Remains to handle $\alpha_2(s)$.

② Have

$$\alpha_2(s) = \int_N^\infty u^{-s}\, d\,\big(A(u) - A(N)\big) = s\int_N^\infty \big(A(u) - A(N)\big)\, u^{-s-1}\, du .$$

Then whenever $\theta > \max\{0, \sigma_c\}$ have $A(u) - A(N) \ll u^\theta$,

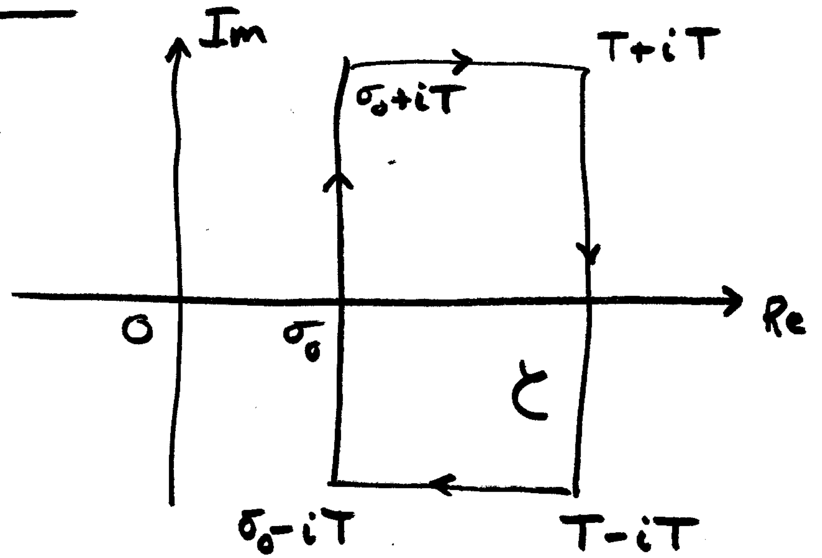
whence for $\sigma > \theta$,

$$\alpha_2(s) \ll |s| \int_N^\infty u^{\theta - \sigma - 1}\, du \ll \frac{|s|}{\sigma - \theta} N^{\theta - \sigma} .$$

Estimate $\displaystyle\int_{\sigma_0 - iT}^{\sigma_0 + iT} \alpha_2(s) \frac{x^s}{s}\, ds$ for

large T by considering

$$\int_{\mathcal{C}} \alpha_2(s) \frac{x^s}{s}\, ds = 0 .$$



So

$$\int_{\sigma_0 - iT}^{\sigma_0 + iT} \alpha_2(s) \frac{x^s}{s}\, ds = \left(-\int_{\sigma_0 + iT}^{T + iT} + \int_{\sigma_0 - iT}^{T - iT} + \int_{T - iT}^{T + iT} \right) \alpha_2(s) \frac{x^s}{s}\, ds$$

3

We have
$$\int_{\sigma_0 \pm iT}^{T \pm iT} \alpha_2(s)\frac{x^s}{s}\,ds \ll \frac{N^{\theta}}{\sigma_0-\theta}\int_{\sigma_0}^{T}\left(\frac{x}{N}\right)^{\sigma} d\sigma \ll \frac{N^{\theta}}{\sigma_0-\theta}\cdot\frac{(x/N)^{\sigma_0}}{\log(x/N)}$$

and
$$\int_{T-iT}^{T+iT} \alpha_2(s)\frac{x^s}{s}\,ds \ll \frac{N^{\theta-T}}{T-\theta}\int_{-T}^{T} x^{T}\,dy \ll N^{\theta}\left(\frac{x}{N}\right)^{T}\cdot\cancel{\frac{T}{T-\theta}}$$

$2x < N$

$$\ll N^{\theta}(x/N)^{\sigma_0}.$$

Thus,
$$\int_{\sigma_0-iT}^{\sigma_0+iT} \alpha_2(s)\frac{x^s}{s}\,ds \ll \frac{N^{\theta}(x/N)^{\sigma_0}}{\sigma_0-\theta} \ll x^{\sigma_0}N^{\theta-\sigma_0}.$$

provided that $\sigma_0 > \theta > \max\{0, \sigma_c\}$.

On combining the contributions from $\alpha_1(s)$ & $\alpha_2(s)$, get
$$\limsup_{T\to\infty}\left|\sum_{n\le x}{}' a_n - \frac{1}{2\pi i}\int_{\sigma_0-iT}^{\sigma_0+iT}\alpha(s)\frac{x^s}{s}\,ds\right| \ll x^{\sigma_0}N^{\theta-\sigma_0}$$

$$\longrightarrow 0 \quad \text{as} \quad N \longrightarrow \infty .$$

Thus
$$\sum_{n \le x}{}' a_n = \lim_{T \to \infty} \frac{1}{2\pi i} \int_{\sigma_0 - iT}^{\sigma_0 + iT} \alpha(s) \frac{x^s}{s} \, ds.$$

<u>Sine integral</u>:
$$\mathrm{si}(x) = - \int_x^{\infty} \frac{\sin u}{u} \, du .$$

By integrating by parts, have for $x \geq 1$, one has
$$\mathrm{si}(x) \ll \min\{1, 1/x\}.$$

Also (evaluate a contour integral), one has
$$\mathrm{si}(x) + \mathrm{si}(-x) = - \int_{-\infty}^{\infty} \frac{\sin u}{u} \, du = -\pi .$$

⑤

<u>**Theorem 9.2.**</u> Suppose that $\sigma_0 > \max\{0, \sigma_a\}$ and $x > 0$. Then

$$\sum_{n \le x}{}' a_n = \frac{1}{2\pi i} \int_{\sigma_0 - iT}^{\sigma_0 + iT} \alpha(s) \frac{x^s}{s} \, ds + R(T),$$

where

$$R(T) = \frac{1}{\pi} \sum_{\frac{x}{2} < n < x} a_n \operatorname{si}\left(T \log \frac{x}{n}\right) - \frac{1}{\pi} \sum_{x < n < 2x} a_n \operatorname{si}\left(T \log \frac{n}{x}\right)$$

$$+ O\left(\frac{4^{\sigma_0} + x^{\sigma_0}}{T} \sum_{n=1}^{\infty} \frac{|a_n|}{n^{\sigma_0}} \right).$$

"Error term" $R(T)$ may be simplified to

$$R(T) \ll \sum_{\substack{\frac{x}{2} < n < 2x \\ n \neq x}} |a_n| \min\left\{1, \frac{x}{T|x-n|}\right\} + \frac{4^{\sigma_0} + x^{\sigma_0}}{T} \sum_{n=1}^{\infty} \frac{|a_n|}{n^{\sigma_0}}.$$

⑥ <u>Proof:</u> Since $\alpha(s)$ is absolutely convergent on $[\sigma_0 - iT, \sigma_0 + iT]$,

we get

$$\frac{1}{2\pi i}\int_{\sigma_0 - iT}^{\sigma_0 + iT} \alpha(s)\frac{x^s}{s}\,ds = \sum_n a_n \cdot \frac{1}{2\pi i}\int_{\sigma_0 - iT}^{\sigma_0 + iT}\left(\frac{x}{n}\right)^s \frac{ds}{s}.$$

The desired conclusion follows as a consequence of the formula (when $\sigma_0 > 0$):

$$(*)\quad \frac{1}{2\pi i}\int_{\sigma_0 - iT}^{\sigma_0 + iT} y^s \frac{ds}{s} = \begin{cases} 1 + O(y^{\sigma_0}/T), & \text{when } y \geq 2, \\ 1 + \frac{1}{\pi}\,\mathrm{si}(T\log y) + O(2^{\sigma_0}/T), & 1 \leq y \leq 2, \\ -\frac{1}{\pi}\,\mathrm{si}(T\log(\frac{1}{y})) + O(2^{\sigma_0}/T), & \frac{1}{2} \leq y \leq 1, \\ O(y^{\sigma_0}/T), & \text{when } 0 < y < \frac{1}{2} \end{cases}$$

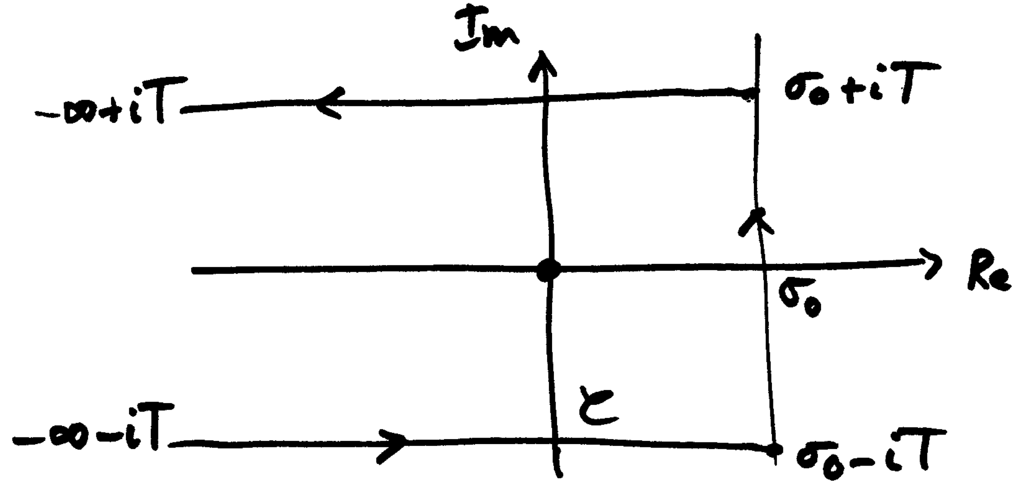
<u>Four cases:</u>

<u>(i) Suppose that $y \geq 2$:</u>

Calculus of residues:

$$\frac{1}{2\pi i}\int_{\mathcal{C}} y^s \frac{ds}{s} = \lim_{s\to 0} y^s = 1$$

simple pole at $s=0$.



⑦ On the other hand, we have

$$\int_{-\infty \pm iT}^{\sigma_0 \pm iT} y^s \frac{ds}{s} = \int_{-\infty}^{\sigma_0} \frac{y^{\sigma \pm iT}}{\sigma \pm iT} d\sigma \ll \frac{1}{T} \int_{-\infty}^{\sigma_0} y^{\sigma} d\sigma = \frac{y^{\sigma_0}}{T \log y}$$

$(y \geq 2)$

$$\ll y^{\sigma_0}/T.$$

Thus

$$\frac{1}{2\pi i} \int_{\sigma_0 - iT}^{\sigma_0 + iT} y^s \frac{ds}{s} = \frac{1}{2\pi i}\left(\int_{\mathcal{C}} - \int_{-\infty - iT}^{\sigma_0 - iT} - \int_{\sigma_0 + iT}^{-\infty + iT} \right) y^s \frac{ds}{s}$$

$$= 1 + O(y^{\sigma_0}/T). \quad \square$$

7 Oct 2020

① **Perron's Formula:**

Recall: $\alpha(s) = \sum_{n=1}^{\infty} a_n n^{-s}$, $\sigma_0 > \max\{0, \sigma_c\}$

$$\sum_{n \le x}{}' a_n = \lim_{T \to \infty} \frac{1}{2\pi i} \int_{\sigma_0 - iT}^{\sigma_0 + iT} \alpha(s) \frac{x^s}{s}\, ds.$$

sine integral : $\mathrm{si}(x) = -\int_x^{\infty} \frac{\sin u}{u}\, du \ll \min\{1, 1/x\}$

$$\mathrm{si}(x) + \mathrm{si}(-x) = -\pi = -\int_{-\infty}^{\infty} \frac{\sin u}{u}\, du$$

Theorem 9.2 Suppose $\sigma_0 > \max\{0, \sigma_a\}$ and $x > 0$. Then

$$\sum_{n \le x}{}' a_n = \frac{1}{2\pi i} \int_{\sigma_0 - iT}^{\sigma_0 + iT} \alpha(s) \frac{x^s}{s}\, ds + R(T),$$

where

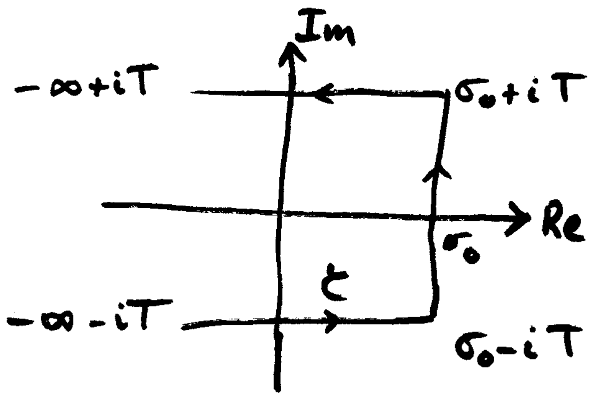
$$R(T) = \frac{1}{\pi} \sum_{\frac{x}{2} < n < x} a_n \,\mathrm{si}\left(T \log \frac{x}{n}\right) - \frac{1}{\pi} \sum_{x < n < 2x} a_n \,\mathrm{si}\left(T \log \frac{n}{x}\right) + O\left(\frac{4^{\sigma_0} + x^{\sigma_0}}{T} \sum_{n=1}^{\infty} \frac{|a_n|}{n^{\sigma_0}}\right).$$

[**Note:** $R(T) \ll \sum_{\substack{\frac{x}{2} < n < 2x \\ n \ne x}} |a_n| \min\left\{1, \frac{x}{T|x-n|}\right\} + \frac{4^{\sigma_0} + x^{\sigma_0}}{T} \sum_{n=1}^{\infty} \frac{|a_n|}{n^{\sigma_0}}.$]

② **Proof:** Use (for $\sigma_0 > 0$)

$$(*) \quad \frac{1}{2\pi i} \int_{\sigma_0 - iT}^{\sigma_0 + iT} y^s \frac{ds}{s} = \begin{cases} 1 + O(y^{\sigma_0}/T), & \text{when } y \geq 2, \\ 1 + \frac{1}{\pi} \operatorname{si}(T \log y) + O(2^{\sigma_0}/T), & \text{when } 1 \leq y \leq 2, \\ -\frac{1}{\pi} \operatorname{si}(T \log(1/y)) + O(2^{\sigma_0}/T), & \text{when } \frac{1}{2} \leq y \leq 1, \\ O(y^{\sigma_0}/T), & \text{when } y \leq \frac{1}{2}. \end{cases}$$

Prove these assertions in turn:



(i) Suppose $y \geq 2$:

Have

$$\frac{1}{2\pi i} \int_{\mathcal{C}} y^s \frac{ds}{s} = 1$$

&

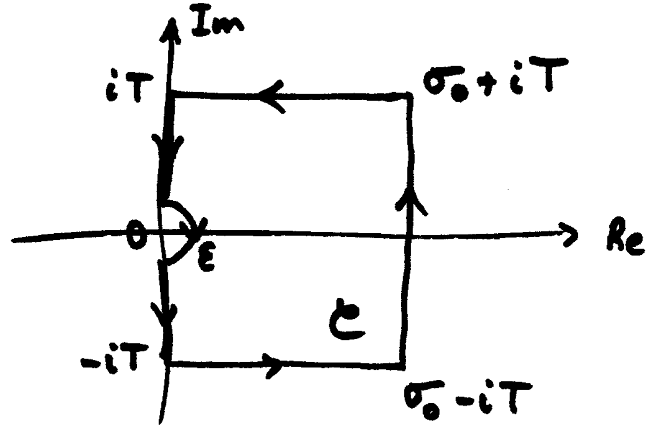
$$\int_{-\infty \pm iT}^{\sigma_0 \pm iT} y^s \frac{ds}{s} \ll \frac{y^{\sigma_0}}{T \log y} \ll \frac{y^{\sigma_0}}{T}$$

(note: $\log y \gg 1$ for $y \geq 2$).

✓

3 (ii) Suppose that $1 \le y \le 2$:

In this case have $\frac{1}{2\pi i}\int_{\mathcal{C}} y^s \frac{ds}{s} = 0.$



We have

$$\int_{\pm iT}^{\sigma_0 \pm iT} y^s \frac{ds}{s} \ll \frac{1}{T}\int_0^{\sigma_0} y^\sigma \, d\sigma \le \frac{1}{T}\int_0^{\sigma_0} 2^\sigma \, d\sigma \ll \frac{2^{\sigma_0}}{T},$$

while the integral over the semi-circular arc of radius ε (suff. small) contributes something asymptotic to

$$\frac{1}{2\pi i}\int_{1/4}^{-1/4} \frac{d(\varepsilon e(\theta))}{\varepsilon e(\theta)} + O(\varepsilon^{*}) \to -\frac{1}{2}, \qquad \text{as } \varepsilon \to 0$$

and

$$\frac{1}{2\pi i}\lim_{\varepsilon\to 0}\left(\int_{i\varepsilon}^{iT} + \int_{-iT}^{-i\varepsilon}\right) y^s \frac{ds}{s} = \frac{1}{2\pi i}\lim_{\varepsilon\to 0}\int_{\varepsilon}^{T} \left(y^{it} - y^{-it}\right)\frac{dt}{t}$$

$$= \frac{1}{\pi}\int_0^{T\log y} \frac{\sin v}{v}\, dv = \frac{1}{\pi}\left(-si(0) + si(T\log y)\right)$$

$$= \frac{1}{2} + \frac{1}{\pi} si(T\log y).$$

4

Thus we deduce

$$\frac{1}{2\pi i}\int_{\sigma_0 - iT}^{\sigma_0 + iT} y^s \frac{ds}{s} = \left(\frac{1}{2} + \frac{1}{\pi}\,\mathrm{si}(T\log y)\right) - \left(-\frac{1}{2}\right) + O\left(\frac{2^{\sigma_0}}{T}\right)$$

$$= 1 + \frac{1}{\pi}\,\mathrm{si}(T\log y) + O\left(\frac{2^{\sigma_0}}{T}\right). \quad \square$$

(iii) Suppose that $\frac{1}{2} \le y \le 1$. The treatment above gives the conclusion desired on noting

$$\begin{aligned}\mathrm{si}\left(-T\log\left(\tfrac{1}{y}\right)\right) &= -\pi - \mathrm{si}\left(T\log\left(\tfrac{1}{y}\right)\right)\\ \| \\ \mathrm{si}(T\log y)\end{aligned}$$

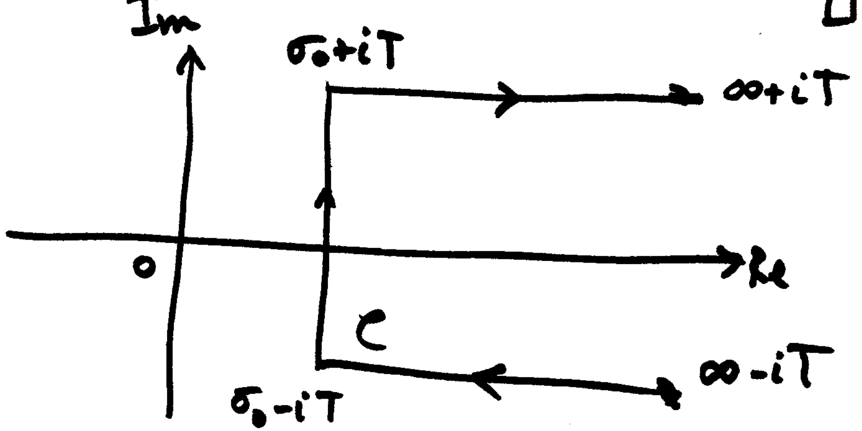
Then

$$\frac{1}{2\pi i}\int_{\sigma_0 - iT}^{\sigma_0 + iT} y^s \frac{ds}{s} = \underbrace{\cancel{1 + (-1)}}_{=0}) - \frac{1}{\pi}\,\mathrm{si}\left(T\log\left(\tfrac{1}{y}\right)\right) + O\left(\frac{2^{\sigma_0}}{T}\right). \quad \square$$

(iv) Suppose that $y \le \frac{1}{2}$.

We flip the contour from (i) around, noting ~~that~~

$$\frac{1}{2\pi i}\int_{\mathcal{C}} y^s \frac{ds}{s} = 0,$$



⑤

$$\int_{\infty \pm iT}^{\sigma_0 \pm iT} y^s \frac{ds}{s} = \int_{\infty}^{\sigma_0} \frac{y^{\sigma \pm iT}}{\sigma \pm iT} \, d\sigma \ll T \int_{\sigma_0}^{\infty} y^{\sigma} d\sigma$$

$$\ll \frac{y^{\sigma_0}}{T \log\left(\frac{1}{y}\right)} \ll y^{\sigma_0}/T \,. \qquad \square$$

To justify the simplification, note ~~that~~

$$\mathrm{si}\left(T \left|\log n/x\right|\right) \ll \min\left\{1, \frac{1}{T\left|\log |n/x|\right|}\right\},$$

But

$$\frac{n}{x} = 1 + \frac{n-x}{x}, \text{ so } \left|\log \frac{n}{x}\right| \asymp \left|\frac{n-x}{x}\right|.$$

Then have

$$(x/2 < n < 2x).$$

$$\mathrm{si}\left(T \left|\log \frac{n}{x}\right|\right) \ll \min\left\{1, \frac{x}{T|x-n|}\right\}$$

Hence

when $\frac{x}{2} < n < 2x$

$$R(T) \ll \sum_{\substack{\frac{x}{2} < n < 2x \\ n \neq x}} |a_n| \min\left\{1, \frac{x}{T|x-n|}\right\} + \frac{4^{\sigma_0} + x^{\sigma_0}}{T} \sum_{n=1}^{\infty} |a_n| n^{-\sigma}. \qquad \square$$

§10. A zero-free region for $\zeta(s)$.

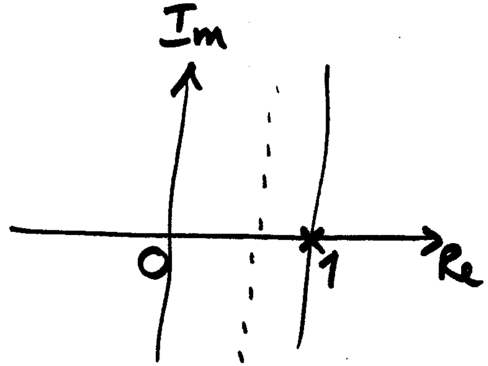
Goal: $$\pi(x) = \int_2^x \frac{dt}{\log t} + O\left(x e^{-c\sqrt{\log x}}\right) \sim \frac{x}{\log x}$$ for some $c > 0$.

$$\downarrow$$

$$\psi(x) = \sum_{n \le x} \Lambda(n) \quad \xleftrightarrow[\text{formula}]{\text{Perron's}} \quad -\frac{\zeta'}{\zeta}(s) = \sum_{n=1}^{\infty} \Lambda(n) n^{-s}.$$

poles of $-\frac{\zeta'}{\zeta}(s)$

$s = 1$ pole of $\zeta(s)$ & $-\frac{\zeta'}{\zeta}(s)$



zeros of $\zeta(s)$ $\longrightarrow$ poles of $-\frac{\zeta'}{\zeta}(s)$?

How many zeros close to a particular point $s \in \mathbb{C}$?

Lemma 10.1 (Jensen's inequality). Suppose that $f(z)$ is analytic in a domain containing the disc $|z| \le R$.

⑦ Suppose also $|f(z)| \le M$ in this disc, and $f(0) \neq 0$. Then, whenever $0 < r < R$, the number of zeros of $f(z)$ in $|z| \le r$ is at most

$$\frac{\log\left(M/|f(0)|\right)}{\log(R/r)}.$$

9 Oct 2020 ① Recall:

§10. A zero-free region for $\zeta(s)$.

Lemma 10·1 (Jensen's inequality) Suppose that $f(z)$ is analytic in a domain containing the disc $|z| \leq R$. Suppose also that $|f(z)| \leq M$ in this disc, and that $f(0) \neq 0$. Then, whenever $0 < r < R$, the number of zeros of $f(z)$ in the disc $|z| \leq r$ is at most

$$\frac{\log(M/|f(0)|)}{\log R/r}.$$

Proof. Let $z_1, \ldots, z_n$ be zeros of $f(z)$ in the disc $|z| \leq R$, counted with multiplicity. We define the Blaschke product

$$g(z) = f(z) \prod_{k=1}^{n} \left(\frac{R^2 - z\bar{z}_k}{R(z - z_k)} \right).$$

Here, the k-th factor has a pole at $z = z_k$, and when $|z| = R$ one has

$$\left| \frac{R^2 - z\bar{z}_k}{R(z - z_k)} \right| = \left| \frac{z(\bar{z} - \bar{z}_k)}{R(z - z_k)} \right| = 1.$$

Then $g(z)$ is analytic in $|z| \leq R$, and when $|z| = R$

② one has

$$|g(z)| = |f(z)| \leq M.$$

Thus, by the maximum modulus principle, we see

$$|f(0)| \prod_{k=1}^{n} \frac{R}{|z_k|} = |g(0)| \leq M.$$

But $|z_k| \leq R$ for every k, and when $|z_k| \leq r$ one has ~~further~~ $R/|z_k| \geq R/r$. Then if the number, say N, of ~~zeros~~ of $f(z)$ in the disc $|z| \leq r$ amongst $z_1, \ldots, z_n$, ~~satisfies~~

$$M \geq |f(0)| \cdot \left(\frac{R}{r}\right)^N,$$

whence

$$N \leq \frac{\log\left(M/|f(0)|\right)}{\log(R/r)}.$$

This is a bound on all the zeros of f in $|z| \leq r$. //

③ **Lemma 10.2.** (Borel - Carathéodory Lemma) Suppose that $h(z)$ is analytic in a domain containing the disc $|z| \le R$. Suppose also that $h(0) = 0$, and that $\operatorname{Re}(h(z)) \le M$ for $|z| \le R$. Then, whenever $|z| \le r < R$, one has

$$|h(z)| \le \frac{2Mr}{R-r} \quad \text{and} \quad |h'(z)| \le \frac{2MR}{(R-r)^2}.$$

Proof: Strategy – expand $h(z)$ and $h'(z)$ as Taylor series about 0, and for this we require estimates for $h^{(k)}(0)/k!$.

Observe that

$$0 = h(0) = \frac{1}{2\pi i} \oint_{|z|=R} h(z) \frac{dz}{z} = \int_0^1 h(Re(\theta))\, d\theta$$

$$\boxed{z = Re(\theta)}$$

Also, when $k > 0$, we have

$$\int_0^1 h(Re(\theta))\, e(+k\theta)\, d\theta = \frac{R^{-k}}{2\pi i} \oint_{|z|=R} h(z) z^{k-1} dz = 0 \quad \text{--- (10.1)}$$

④

$$\int_0^1 h(Re(\theta))\, e(-k\theta)\, d\theta = \frac{R^k}{2\pi i} \oint_{|z|=R} h(z) z^{-k-1}\, dz$$

$$= R^k \frac{h^{(k)}(0)}{k!} \quad \text{——— (10.2)}$$

Combining (10.1) & (10.2) when $k>0$, we obtain (for $\phi \in \mathbb{R}$)

$$\int_0^1 h(Re(\theta)) \left(1 + \cos(2\pi(k\theta+\phi))\right) d\theta$$

$$= \int_0^1 h(Re(\theta)) \left(1 + \frac{e(k\theta+\phi) + e(-k\theta-\phi)}{2}\right) d\theta$$

$$= \frac{e(-\phi)}{2} \cdot R^k \frac{h^{(k)}(0)}{k!}.$$

Thus, since $\mathrm{Re}(h(z)) \le M$, we see ~~that~~

$$\mathrm{Re}\left(\frac{e(-\phi)}{2} \cdot \frac{R^k h^{(k)}(0)}{k!}\right) \le \int_0^1 M\left(1 + \cos(2\pi(k\theta+\phi))\right) d\theta$$

5

We choose ϕ so that $e(-\phi)\, h^{(k)}(0) = |h^{(k)}(0)|$,

whence
$$\frac{|h^{(k)}(0)|}{k!} \leq \frac{2M}{R^k} \int_0^1 1 + \cos(2\pi(k\theta+\phi))\, d\theta = \frac{2M}{R^k}.$$

Apply this estimate in a Taylor expansion — whenever $|z| \leq r < R$, one has

$$|h(z)| \leq \sum_{k=1}^{\infty} \frac{|h^{(k)}(0)|}{k!} r^k \leq 2M \sum_{k=1}^{\infty} \left(\frac{r}{R}\right)^k = \frac{2Mr}{R-r}$$

and
$$|h'(z)| \leq \sum_{k=1}^{\infty} \frac{|h^{(k)}(0)|}{k!} k r^{k-1} \leq 2M \sum_{k=1}^{\infty} k \left(\frac{r}{R}\right)^{k-1} = \frac{2MR}{(R-r)^2}.$$

//

(6)

<u>Lemma 10.3.</u> Suppose that $f(z)$ is analytic in a domain containing the disc $|z| \le 1$. Suppose also that $|f(z)| \le M$ in this disc, and $f(0) \neq 0$. Then, whenever r and R are fixed with $0 < r < R < 1$, and $|z| \le r$, one has

$$\frac{f'}{f}(z) = \sum_{k=1}^{n} \frac{1}{z - z_k} + O_{r,R}\left(\log\left(\frac{M}{|f(0)|}\right)\right),$$

where the summation is over all zeros of f with $|z_k| \le R$.

<u>Proof.</u> If necessary, we can suppose that $f(z) \neq 0$ for $|z| = R$, for we can replace R by $R + \varepsilon$ for arbitrarily small $\varepsilon > 0$. Define

$$g(z) = f(z) \prod_{k=1}^{n} \frac{R^2 - z\bar{z}_k}{R(z - z_k)}.$$

Note, as a consequence of Lemma 10.1,

$$n \le \frac{\log(M/|f(0)|)}{\log(1/R)} \ll_R \log\left(\frac{M}{|f(0)|}\right). \quad \text{—(10.}$$

Also, when $|z| = R$, one has
$$\left| \frac{R^2 - z\bar{z}_k}{R(z - z_k)} \right| = 1,$$
Whence for $|z| = R$ one has $|g(z)| = |f(z)| \leq M$.
Then from max. mod. principle, have $|g(z)| \leq M$ for $|z| \leq R$.

Observe next ~~that~~
$$|g(0)| = |f(0)| \prod_{k=1}^{n} \frac{R}{|z_k|} \geq |f(0)| > 0.$$
Since $g(z)$ has no zeros in $|z| \leq R$, we can put
$$h(z) = \log\left(\frac{g(z)}{g(0)}\right).$$
Then $h(0) = 0$, and when $|z| \leq R$,
$$\begin{aligned} \mathrm{Re}(h(z)) &= \log|g(z)| - \log|g(0)| \\ &\leq \log M - \log|f(0)| = \log\left(\frac{M}{|f(0)|}\right). \end{aligned}$$

⑧ Now apply Borel - Carathéodory lemma : when $|z| \le r$ have

$$|h'(z)| \ll_{r,R} \log\left(\frac{M}{|f(0)|}\right).$$

But

$$h'(z) = \frac{g'}{g}(z) = \frac{f'}{f}(z) - \sum_{k=1}^{n} \frac{1}{z - z_k} + \sum_{k=1}^{n} \frac{1}{z - R^2/\bar{z}_k}$$

12 Oct 2020 | Recall: Jensen's inequality: (Lemma 10.1)

①

f analytic in $|z| \le R$ with $|f(z)| \le M$ & $f(0) \neq 0$.

Then for $0 < r < R$, $\#(f(z) = 0 : |z| \le r) \le \dfrac{\log(M/|f(0)|)}{\log(R/r)}$

Borel - Carathéodory Lemma (Lemma 10.2)

$h(z)$ analytic in $|z| \le R$ with $\operatorname{Re}(h(z)) \le M$ & $h(0) = 0$.

Then for $|z| \le r < R$, have $|h(z)| \le \dfrac{2Mr}{R-r}$ & $|h'(z)| \le \dfrac{2MR}{(R-r)^2}$.

Lemma 10.3 Suppose that $f(z)$ is analytic in a domain containing the disc $|z| \le 1$. Suppose also that $|f(z)| \le M$ in this disc, and that $f(0) \neq 0$. Then whenever r and R are fixed with $0 < r < R < 1$, and $|z| \le r$, one has

$$\frac{f'}{f}(z) = \sum_{k=1}^{n} \frac{1}{z - z_k} + O_{r,R}\left(\log\left(\frac{M}{|f(0)|}\right)\right),$$

where the summation is over all zeros of f for which $|z_k| \le R$.

Proof. (so far) Define Blaschke product

$$g(z) = f(z) \prod_{k=1}^{n} \left(\frac{R^2 - z\bar{z}_k}{R(z - z_k)} \right).$$

②

Lemma 10·1 $\longrightarrow$ $$n \le \frac{\log(M/|f(0)|)}{\log(1/R)} \ll_{R,r} \log\left(\frac{M}{|f(0)|}\right) \quad \text{———} \quad (10\cdot 8)$$

- $\underset{\overset{\shortparallel}{|f(z)|}}{|g(z)|} \le M$ for $|z| \le 1$ (maximum modulus principle)
- $|g(0)| \ge |f(0)| > 0.$

Define $$h(z) = \log\left(\frac{g(z)}{g(0)}\right)$$ (OK since g has no zeros in $|z| \le R$)

- $h(0) = 0$
- $\mathrm{Re}(h(z)) = \log|g(z)| - \log|g(0)| \le \log M - \log|f(0)| = \log\left(\frac{M}{|f(0)|}\right)$

Borel - Carathéodory lemma shows, when $|z| \le r$, one has

$$h'(z) \ll_{r,R} \log\left(\frac{M}{|f(0)|}\right)$$

$$\shortparallel$$

$$\frac{g'}{g}(z) = \frac{f'}{f}(z) - \sum_{k=1}^{n} \frac{1}{z - z_k} + \sum_{k=1}^{n} \frac{1}{z - R^2/\bar{z}_k}.$$

Thus

$$\frac{f'}{f}(z) - \sum_{k=1}^{n} \frac{1}{z - z_k} + \sum_{k=1}^{n} \frac{1}{z - R^2/\bar{z}_k} \ll_{R,r} \log\left(\frac{M}{|f(0)|}\right).$$

③ Notice that $|R^2/\bar{z}_k| \geq R$, so that when $|z| \leq r$,

$|z_k| \leq R$

one has $|z - R^2/\bar{z}_k| \geq R - r$, whence

$$\sum_{k=1}^{n} \frac{1}{z - R^2/\bar{z}_k} \leq \frac{n}{R-r} \overset{(10.3)}{\ll_{R,r}} \log\left(\frac{M}{|f(0)|}\right)$$

Thus

$$\frac{f'}{f}(z) - \sum_{k=1}^{n} \frac{1}{z - z_k} \ll_{R,r} \log\left(\frac{M}{|f(0)|}\right).$$

//

Need bounds for $\zeta(s)$ for $s \in \mathbb{C}$.

<u>Lemma 10.4</u> Let $\delta > 0$ be fixed. Then one has

$$\zeta(s) = \frac{1}{s-1} + O_\delta(1)$$

uniformly for s in the rectangle $\delta \leq \sigma \leq 2$ and $|t| \leq 1$, and further

$$\zeta(s) \ll_\delta (1 + \tau^{1-\sigma}) \min\left\{ \frac{1}{|\sigma - 1|}, \log \tau \right\}$$

uniformly for $\delta \le \sigma \le 2$ and $|t| \ge 1$. $[\tau := |t| + 4]$.

<u>Proof.</u> First claim — recall that

$$\zeta(s) = \frac{s}{s-1} - s\int_1^\infty \frac{\{u\}}{u^{s+1}}\,du \qquad (\sigma > 0) \qquad (\text{see } (3\cdot3))$$

The second term here is $\ll |s| \int_1^\infty u^{-1-\sigma}\,du \ll \frac{|s|}{\sigma} \ll \frac{1}{\delta}$

when $\delta \le \sigma \le 2$ and $|t| \le 1$. □

Turn to second assertion. Recall Theorem 3·5, namely that when $s \ne 1$ and $\sigma > 0$, $x > 0$, one has

$$\zeta(s) = \sum_{n \le x} n^{-s} + \frac{x^{1-s}}{s-1} + \frac{\{x\}}{x^s} - s\int_x^\infty \frac{\{u\}}{u^{s+1}}\,du.$$

The integral here satisfies

$$\int_x^\infty \frac{\{u\}}{u^{s+1}}\,du \ll \int_x^\infty u^{-1-\sigma}\,du \ll \frac{x^{-\sigma}}{\sigma},$$

whence

$$\zeta(s) = \sum_{n \le x} n^{-s} + \frac{x^{1-s}}{s-1} + O\left(\frac{|s|}{\sigma} x^{-\sigma}\right)$$

5

$$\zeta(s) = \sum_{n \le x} n^{-s} + \frac{x^{1-s}}{s-1} + O_s\left(\tau x^{-\sigma}\right). \quad \text{—— (10·4)}$$

$(s = \sigma + it)$

Can bound the sum here by

$$\sum_{n \le x} n^{-s} \ll \sum_{n \le x} n^{-\sigma} \ll 1 + \int_1^x u^{-\sigma}\,du \qquad (\sigma \geqslant 0).$$

When $|\sigma - 1| \le 1/\log x$ and $1 \le u \le x$, one has $u^{-\sigma} \asymp u^{-1}$, and then $\int_1^x u^{-\sigma}\,du \ll \int_1^x u^{-1}\,du = \log x$.

Meanwhile, when $\sigma - 1 > 1/\log x$, one has instead

$$\int_1^x u^{-\sigma}\,du < \int_1^\infty u^{-\sigma}\,du = \frac{1}{\sigma - 1},$$

and when $0 \leqslant \sigma \leqslant 1 - 1/\log x$,

$$\int_1^x u^{-\sigma}\,du = \frac{x^{1-\sigma} - 1}{1 - \sigma} < \frac{x^{1-\sigma}}{1-\sigma}.$$

Hence

$$\sum_{n \le x} n^{-s} \ll \left(1 + x^{1-\sigma}\right) \min\left\{ \frac{1}{|1-\sigma|}, \log x \right\}.$$

⑥ uniformly for $0 \le \sigma \le 2$.

Substitute into (10.4) to get, with $x = \tau$,

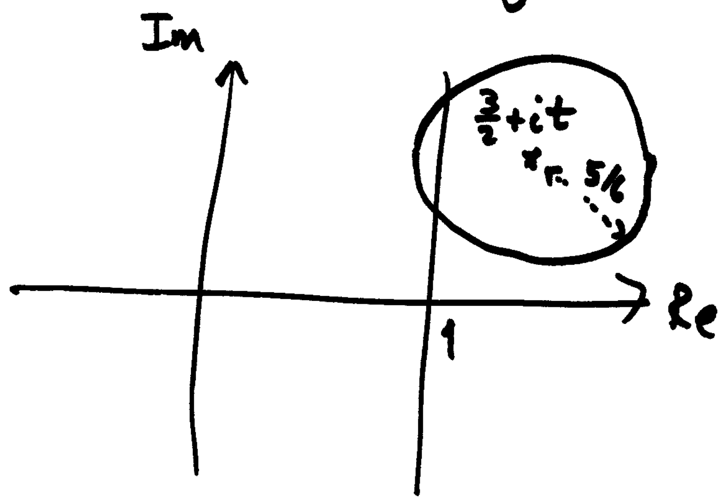
$$\zeta(s) \ll (1 + \tau^{1-\sigma}) \min\left\{\frac{1}{|\sigma - 1|}, \log \tau\right\}. \qquad \square$$

<u>Lemma 10.5.</u> Suppose that $|t| \geq 7/8$ and $\frac{5}{6} \le \sigma \le 2$.

Then one has

$$\frac{\zeta'}{\zeta}(s) = \sum_{\rho} \frac{1}{s-\rho} + O\Big(\log \underbrace{(|t| + 4)}_{\tau}\Big),$$

where the summation over ρ is over all zeros of $\zeta(s)$ with $\left|\rho - \left(\frac{3}{2} + it\right)\right| \le \frac{5}{6}$.



<u>Proof:</u> We apply Lemma 10.3 to

$$f(z) := \zeta\Big(\underbrace{z + \left(\tfrac{3}{2} + it\right)}_{s}\Big),$$

with $R = 5/6$ and $r = 2/3$.

⑦. We have

$$f(0) = \zeta\left(\tfrac{3}{2}+it\right) = \prod_p \left(1-p^{-\frac{3}{2}-it}\right)^{-1}$$

$$\Rightarrow \quad |f(0)| \geqslant \prod_p \left(1-p^{-3/2}\right)^{-1} > 0 \quad (\text{abs. conv. for } \mathrm{Re}(s)>1).$$

Thus $|f(0)| \gg 1$. Also, by Lemma 10.4, we have

$$|f(z)| \ll |\zeta(z+(\tfrac{3}{2}+it)| \ll \tau^{1/2}\log\tau \ll \tau$$

for $|z| \le 1$.

Thus,

$$\frac{f'}{f}(z) = \sum_\rho \frac{1}{(z+\frac{3}{2}+it)-\rho} + O(\log\tau)$$

$$\Rightarrow \quad \frac{\zeta'}{\zeta}(s) = \sum_\rho \frac{1}{s-\rho} + O(\log\tau). \quad //$$

14 Oct 2020

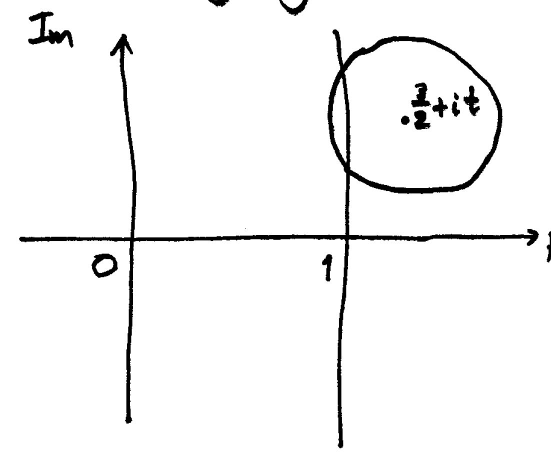
① Recall: Lemma 10.5 Suppose that $|t| \geq \frac{7}{8}$ and $\frac{5}{6} \leq \sigma \leq 2$.

Then one has

$$\frac{\zeta'}{\zeta}(s) = \sum_{\rho} \frac{1}{s-\rho} + O(\log(\underbrace{|t|+4}_{\tau})),$$

where the summation is over all zeros ρ of $\zeta(s)$ satisfying

$$|\rho - (\tfrac{3}{2}+it)| \leq \tfrac{5}{6}.$$



Next — ? how to find a zero-free region for $\zeta(s)$?

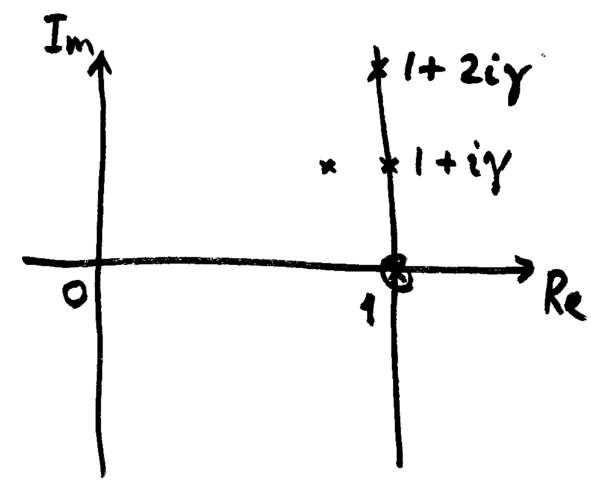
Thought experiment:

Consider $\delta \to 0+$, and $s = 1+\delta+i\gamma$.

If $s = 1+i\gamma$ were a zero of $\zeta(s)$, then

$$\frac{\zeta'}{\zeta}(1+\delta+i\gamma) \sim \frac{m}{\delta},$$

where m is the multiplicity of this zero.



Notice here that

(2)

$$\mathrm{Re}\left(\frac{\zeta'}{\zeta}(1+\delta+i\gamma)\right) = -\sum_{n=1}^{\infty} \Lambda(n)\, n^{-1-\delta} \cos(\gamma \log n)$$

$$\leq \sum_{n=1}^{\infty} \Lambda(n)\, n^{-1-\delta} = -\frac{\zeta'}{\zeta}(1+\delta) \sim \frac{1}{\delta},$$

$$\Downarrow$$

$$m \leq 1.$$

So this can happen only when $\cos(\gamma \log n) \approx -1$ for "most" $n = p^h$, whence $p^{i\gamma} \approx -1$ for most primes p.

$$\downarrow$$

$$p^{2i\gamma} = (p^{i\gamma})^2 \approx +1 \qquad ''$$

$$\Rightarrow \mathrm{Re}\left(\frac{\zeta'}{\zeta}(1+\delta+2i\gamma)\right) \sim -\sum_{n=1}^{\infty} \Lambda(n)\, n^{-1-\delta} \sim -\frac{1}{\delta}$$

(pole at $1+2i\gamma$) distinct from pole at $s=1$.

③

<u>Lemma 10.6</u> When $\sigma > 1$, one has

$$\mathrm{Re}\left(-3\frac{\zeta'}{\zeta}(\sigma) - 4\frac{\zeta'}{\zeta}(\sigma+it) - \frac{\zeta'}{\zeta}(\sigma+2it)\right) \geqslant 0.$$

<u>Proof.</u> The lhs is

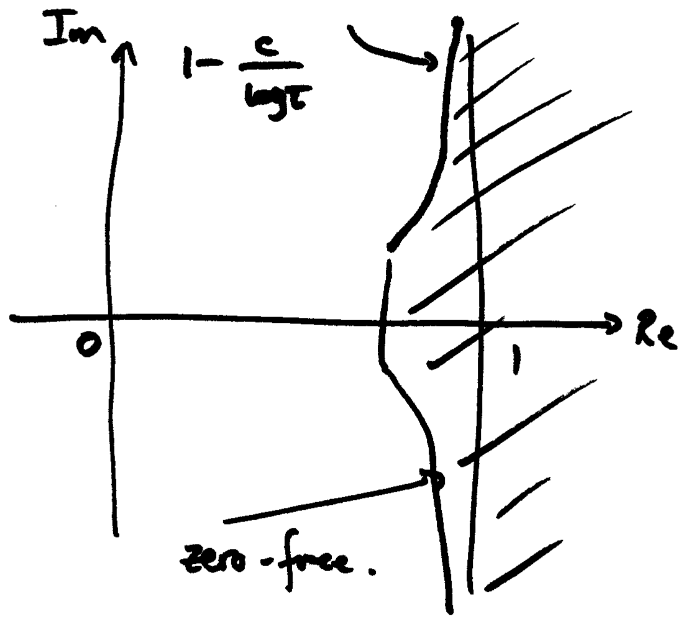
$$\sum_{n=1}^{\infty} \Lambda(n)\, n^{-\sigma} \big(\underbrace{3 + 4\cos(t\log n) + \cos(2t\log n)}_{=\,2(1+\cos(t\log n))^2 \,\geqslant\, 0}\big)$$

for all t. //

$$\cos 2\theta = 2\cos^2\theta - 1$$

④

Theorem 10.7. There is an absolute constant $c > 0$ such that $\zeta(s) \neq 0$ for $\sigma \geq 1 - c/\log\tau$.



World Record: Korobov, 1958
Vinogradov, 1958

$$\sigma \geq 1 - c(\log\tau)^{-2/3}(\log\log\tau)^{-1/3}.$$

Riemann Hypothesis: $\sigma > \frac{1}{2}$.

Proof: When $\sigma > 1$, one has

$$|\zeta(s)| = \left|\prod_p (1-p^{-s})^{-1}\right| \geq \prod_p (1-p^{-\sigma})^{-1} > 0.$$

Then $\zeta(s) \neq 0$ for $\sigma > 1$. $\square$

Next, from formula $\zeta(s) = \frac{s}{s-1} - s\int_1^\infty \frac{\{u\}}{u^{s+1}}\,du$, we find

$$\left|\zeta(s) - \frac{s}{s-1}\right| \leq |s|\int_1^\infty \frac{du}{u^{\sigma+1}} = \frac{|s|}{\sigma} \quad (\sigma > 0)$$

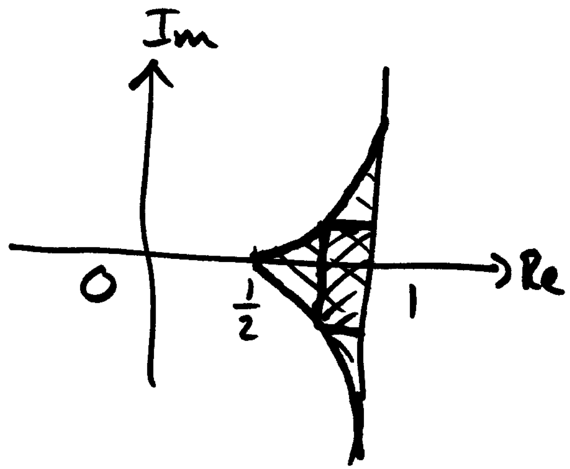
Thus $\zeta(s) \neq 0$ when $\sigma > |s-1|$.

This is satisfied when $\sigma^2 > (\sigma-1)^2 + t^2$

$$\iff \sigma > (1+t^2)/2.$$

$$\boxed{\text{Then } \zeta(s) \neq 0 \text{ for } |t| \le \tfrac{7}{8} \text{ \& } \tfrac{8}{9} \le \sigma \le 1}$$

$$\left(\tfrac{1}{2}\left(\left(\tfrac{7}{8}\right)^2 + 1\right) = 1 - \tfrac{15}{128} < \tfrac{8}{9} \right).$$



Suppose next that there is a zero $\rho_0 = \beta_0 + i\gamma_0$ of $\zeta(s)$ with $\frac{5}{6} \le \beta_0 \le 1$ and $|\gamma_0| \ge \frac{7}{8}$.

Since $\mathrm{Re}(\rho) \le 1$ for all zeros of $\zeta(s)$, so

$$(*) \qquad \mathrm{Re}\left(\frac{1}{s-\rho}\right) > 0 \quad \text{whenever } \sigma > 1.$$

We apply Lemma 10.5 with $s = 1 + \delta + i\gamma_0$ to see that for small positive values of δ. For a suitable

⑥ $c_1 > 0$, one has

$$-\operatorname{Re}\left(\frac{\zeta'}{\zeta}(1+\delta+i\gamma_0)\right) = \operatorname{Re}\left(-\sum_{\rho}\frac{1}{s-\rho} + O(\log \tau)\right)$$

$$\overset{(*)}{\leq} \frac{-1}{1+\delta-\beta_0} + c_1 \log(|\gamma_0|+4).$$

Similarly, but now with $s = 1+\delta+2i\gamma_0$,

$$-\operatorname{Re}\left(\frac{\zeta'}{\zeta}(1+\delta+2i\gamma_0)\right) \overset{(*)}{\leq} c_1 \log(|2\gamma_0|+4).$$

Finally, we have

$$-\frac{\zeta'}{\zeta}(1+\delta) = \frac{1}{\delta} + O(1)$$

Hence, combining via Lemma 10.6, we discern that for a suitable $c_2 > 0$, one has

$$\frac{3}{\delta} - \frac{4}{1+\delta-\beta_0} + c_2 \log(|\gamma_0|+4) \geq 0.$$

⑦

Put $\delta = \dfrac{1}{2c_2 \log(|\gamma_0|+4)}$, so ~~that~~

$$7 c_2 \log(|\gamma_0|+4) \geq 4/(1+\delta-\beta_0)$$

$$\Downarrow$$

$$1+\delta-\beta_0 = 1 + \frac{1}{2c_2\log(|\gamma_0|+4)} - \beta_0$$

$$\geq \frac{4}{7c_2\log(|\gamma_0|+4)}$$

$$\Rightarrow \quad 1-\beta_0 \geq \frac{1}{14 c_2 \log(|\gamma_0|+4)} = \frac{1}{14c_2\log\tau}.$$ //

$\zeta(s) \ll \log\tau$ $\qquad$ $1/\zeta(s) \ll \log\tau$

$\frac{\zeta'}{\zeta}(s) \ll \log\tau$ $\qquad$ $|\log\zeta(s)| \leq \log\log\tau + O(1)$

16 Oct 2020

①

Problem Sheet 2.

$$\tau^*(n) = \sum_{d^2 \mid n} 1 = \sum_{ed^2 = n} 1$$

Q1 (i) When $\sigma > 1$, one has

$$\zeta(s)\zeta(2s) = \left(\sum_{n=1}^{\infty} n^{-s}\right)\left(\sum_{m=1}^{\infty} (m^2)^{-s}\right) = \sum_{k=1}^{\infty} \sum_{k=nm^2} k^{-s} = \sum_{k=1}^{\infty} \tau^*(k) k^{-s}.$$

$\square$

(ii)

$$\sum_{1 \le n \le x} \tau^*(n) = \sum_{1 \le n \le x} \sum_{d^2 \mid n} 1 = \sum_{1 \le d \le x^{1/2}} \sum_{1 \le m \le x/d^2} 1$$

$n = md^2$

$$= \sum_{1 \le d \le x^{1/2}} \left(\frac{x}{d^2} + O(1) \right) = x \sum_{d=1}^{\infty} \frac{1}{d^2} - x \sum_{d > x^{1/2}} \frac{1}{d^2} + O(x^{1/2})$$

$$= x\zeta(2) + O(x^{1/2}). \quad \square$$

Q2 (i) ? $\pi(x) \asymp x/\log x$? $\pi(x) = \sum_{p \le x} 1$

$$\sum_{p \le x} \frac{\log p}{p} = \log x + O(1) \; ; \; \sum_{p \le x} \frac{1}{p} = \log\log x + O(1).$$

② When $\delta > 0$

$$\sum_p \frac{1}{p(\log p)^\delta} = \int_{2^-}^{\infty} (\log u)^{-1-\delta}\, d\,A(u), \quad \text{where } A(u) = \sum_{p \le u} \frac{\log p}{p} = \log u + O(1)$$

$$= \Big[\underset{\log u + O(1)}{\underset{\parallel}{A(u)}} (\log u)^{-1-\delta} \Big]_{2^-}^{\infty} + \int_{2^-}^{\infty} (1+\delta)(\log u)^{-2-\delta} \frac{A(u)}{u}\, du$$

$$= 0 + (1+\delta) \int_{2^-}^{\infty} (\log u)^{-1-\delta} \frac{du}{u} + O\Big(\int_{2^-}^{\infty} (\log u)^{-2-\delta} \frac{du}{u} \Big)$$

$$\ll 1$$

$$\boxed{\int_2^{\infty} \frac{dt}{t(\log t)^\theta} < \infty \text{ for } \theta > 1}$$

(ii) Similarly,

$$\sum_{p \le x} \frac{1}{p \log\log p} = \frac{1}{2 \log\log 2} + \underbrace{\int_{3^-}^{x} (\log u)^{-1} (\log\log u)^{-1}\, d\,A(u)}_{I}$$

$$I = \Big[A(u) (\log u)^{-1} (\log\log u)^{-1} \Big]_{3^-}^{x} + \int_{3^-}^{x} \Big(\frac{1}{u(\log u)^2 \log\log u} + \frac{1}{u(\log u)^2 (\log\log u)^2} \Big) \times A(u)\, du.$$

③

$$= \int_{3^-}^{x} \frac{\log u + O(1)}{u(\log u)^2 \log\log u}\,du + O\left(\cancel{\frac{1}{\log\log x}}\right) + O(1)$$

$$= \int_{3^-}^{x} \frac{du}{u\log u \log\log u} + O(1)$$

$A(u) \ll \log u \Rightarrow \int_3^\infty \frac{A(u)\,du}{u(\log u)^2(\log\log u)^2}$

$\ll \int_3^\infty \frac{du}{u\log u(\log\log u)^2} < \infty$

$$I = \log\log\log x - \log\log\log 3 + O(1)$$

& $\int_{3^-}^\infty \frac{du}{u(\log u)^2\log\log u} < \infty$

Thus

$$\sum_{p\le x} \frac{1}{p\log\log p} = \log\log\log x + O(1). \quad \square$$

④ Q3 (i) When $(m,n)=1$, one has that either m or n is odd, or both, whence

$$mn-1 = (m-1)(n-1) + (m-1) + (n-1) \equiv (m-1)+(n-1) \pmod 2$$

$$\Rightarrow (-1)^{mn-1} = (-1)^{m-1}\cdot(-1)^{n-1} \quad \text{when } (m,n)=1.$$

$f(1) = 1.$ ✓ □

When $\sigma > 1$, use absolute convergence of $\sum_{n=1}^{\infty}(-1)^{n-1}n^{-s}$ to obtain

$$\sum_{n=1}^{\infty}(-1)^{n-1}n^{-s} = \sum_{n=1}^{\infty}\prod_{p^h \| n}(-1)^{p^h-1}p^{-hs}$$

$$= \left(1-\sum_{h=1}^{\infty}2^{-hs}\right)\prod_{p>2}\left(1+\sum_{h=1}^{\infty}p^{-hs}\right)$$

$$= \frac{1-2^{-s}/(1-2^{-s})}{(1-2^{-s})^{-1}}\underbrace{\prod_{p}(1-p^{-s})^{-1}}_{\zeta(s)}$$

$$= (1-2^{1-s})\zeta(s)$$

⑤ Notice that $(1-2^{1-s})\zeta(s)$ is analytic for $\sigma > 0$ except possibly at $s=1$, where $\zeta(s)$ has a simple pole. But
$$1-2^{1-s} = 1-e^{(1-s)\log 2}$$
$$= (s-1)\log 2 + \cdots$$
has a zero at $s=1$, which kills the simple pole of $\zeta(s)$. So $(1-2^{1-s})\zeta(s)$ is analytic for $\mathrm{Re}(s) > 0$, and hence equal to $\sum_{n=1}^{\infty} (-1)^{n-1} n^{-s}$ (since $\sigma_c = 0$).

(ii) By considering Laurent series:

$$(1-2^{1-s})\zeta(s) = \left((s-1)\log 2 - \frac{(s-1)^2}{2!}(\log 2)^2 + \cdots \right)$$
$$\cdot \left(\frac{1}{s-1} + c_0 + c_1(s-1) + \cdots \right)$$

⑥

$$= \log 2 + \left(c_0 \log 2 - \tfrac{1}{2}(\log 2)^2 \right)(s-1) + c_1'(s-1)^2 + \dots$$

Then

$$\sum_{n=1}^{\infty} (-1)^{n-1}/n = \lim_{s \to 1} (1 - 2^{1-s})\zeta(s) = \log 2$$

$$\sum_{n=1}^{\infty} (-1)^{n} \log n/n = \lim_{s \to 1} \frac{d}{ds}\left(\sum_{n=1}^{\infty} (-1)^{n-1}/n^s \right)$$

$$= \lim_{s \to 1} \frac{d}{ds}\left((1 - 2^{1-s})\zeta(s) \right)$$

$$= c_0 \log 2 - \tfrac{1}{2}(\log 2)^2 .$$

19 Oct 2020

①

Recall: **Theorem 10.7.** There is an absolute constant $c>0$ such that $\zeta(s)\neq 0$ for $\sigma \geqslant 1-\dfrac{c}{\log \tau}$.

(remember that $\tau := |t|+4$).

Theorem 10.8. Suppose that $c>0$ is an absolute constant with the property that $\zeta(s)\neq 0$ for $\sigma \geqslant 1 - c/\log\tau$. Then whenever

$\sigma > 1-\frac{1}{2}c/\log\tau$ and $|t|\geqslant 7/8$, one has

$$\zeta(s) \ll \log\tau, \qquad \frac{\zeta'}{\zeta}(s) \ll \log\tau, \qquad \frac{1}{\zeta(s)} \ll \log\tau, \qquad (10.5)$$

and

$$|\log \zeta(s)| \leq \log\log\tau + O(1). \qquad (10.6)$$

Proof. The first bound in (10.5) follows from Lemma 10.4, since for $\sigma \geqslant 1-\frac{1}{2}c/\log\tau$, one has

$$\begin{aligned}\zeta(s) &\ll (1+\tau^{1-\sigma})\min\left\{\frac{1}{(\sigma-1)}, \log\tau\right\} \qquad \text{for } |t|\geqslant 1\\ &\ll \log\tau,\end{aligned}$$

whilst $\zeta(s) = \dfrac{1}{s-1} + O(1) \ll \log\tau$, for $\frac{7}{8}\leq |t| \leq 1$.

(2) Consider the 2nd estimate in (10.5). When $\sigma > 1$, one has

$$\left| \frac{\zeta'}{\zeta}(s) \right| \leq \sum_{n=1}^{\infty} \Lambda(n) n^{-\sigma} = -\frac{\zeta'}{\zeta}(\sigma) \ll \frac{1}{\sigma - 1}$$

(using Laurent series for $\zeta(s)$). Thus, whenever $\sigma \geqslant 1 + 1/\log\tau$, it follows that $\frac{\zeta'}{\zeta}(s) \ll \log\tau$. We now extrapolate from this bound for $\frac{\zeta'}{\zeta}(s_1)$ when $s_1 = 1 + \frac{1}{\log\tau} + it$, to obtain bounds for $\frac{\zeta'}{\zeta}(s)$ inside the line $\sigma = 1$.

We have

$$\frac{\zeta'}{\zeta}(s_1) \ll \log\tau ,$$

and further, from Lemma 10.5, whenever $|t| \geq 7/8$ and $1 - \frac{1}{2}c/\log\tau \leq \sigma \leq 1 + 1/\log\tau$,

$$\frac{\zeta'}{\zeta}(s) = \sum_{\rho} \frac{1}{s - \rho} + O(\log\tau),$$ where the sum is over zeros ρ with $|\rho - (\frac{3}{2} + it)| \leq 5/6$.

Thus

$$\log \tau \gg \frac{\zeta'}{\zeta}(s_1) = \sum_{\rho} \frac{1}{s_1-\rho} + O(\log \tau)$$

$$\Rightarrow \mathrm{Re}\left(\sum_{\rho} \frac{1}{s_1-\rho} \right) \ll \log \tau . \qquad (*)$$

Given any $s = \sigma + it$ with $1 - \frac{1}{2} c/\log\tau \le \sigma \le 1 + \frac{1}{\log\tau}$, one ~~therefore~~ has

$$\frac{\zeta'}{\zeta}(s) - \frac{\zeta'}{\zeta}(s_1) = \sum_{\rho} \left(\frac{1}{s-\rho} - \frac{1}{s_1-\rho} \right) + O(\log\tau)$$

But

$$\frac{1}{s-\rho} - \frac{1}{s_1-\rho} = \frac{s_1-s}{(s-\rho)(s_1-\rho)} \ll \frac{1}{|s_1-\rho|^2 \cdot \log\tau}$$

$$(\mathrm{Im}(s) = t = \mathrm{Im}(s_1)).$$

To justify ~~that~~ $|s-\rho| \asymp |s_1-\rho|$, observe: since $\zeta(s) \neq 0$ for $\sigma > 1 - c/\log\tau$, then $\mathrm{Re}(\rho) \le 1 - c/\log\tau$.

④. Thus $\mathrm{Re}(s-\rho) \gg 1/\log\tau$ & $|s-s_1| \ll 1/\log\tau$, one sees ~~that~~ $|s-\rho| \sim |s_1-\rho|$. Then since $|s_1-\rho| \gg 1/\log\tau$, one obtains

$$\frac{1}{s-\rho} - \frac{1}{s_1-\rho} \ll \mathrm{Re}\left(\frac{1}{s_1-\rho}\right),$$

whence

$$\sum_\rho \left(\frac{1}{s-\rho} - \frac{1}{s_1-\rho}\right) \ll \mathrm{Re}\left(\sum_\rho \frac{1}{s_1-\rho}\right) \boxed{\cancel{+O(\log\tau)}} \overset{(*)}{\ll} \log\tau.$$

Thus

$$\frac{\zeta'}{\zeta}(s) = \frac{\zeta'}{\zeta}(s_1) + O(\log\tau) \ll \log\tau. \quad \square$$

The third estimate of (10.5) follows from (10.6) by noting $\log\left(\frac{1}{|\zeta|}\right) = -\mathrm{Re}(\log\zeta) = -\log\log\tau + O(1)$. implies $1/|\zeta(s)| \ll \log\tau$. So concentrate on (10.6).

We move from $\sigma = 1 + 1/\log\tau$ to left of $\sigma = 1$.

Observe ~~that~~ when $\sigma > 1$, one has

$$|\log \zeta(s)| \le \sum_{n=2}^{\infty} \frac{\Lambda(n)}{\log n} n^{-\sigma} = \log \zeta(\sigma).$$

But
$$\zeta(\sigma) = \frac{\sigma}{\sigma-1} - \sigma \int_1^{\infty} \frac{\{u\}}{u^{\sigma+1}}\, du < \frac{\sigma}{\sigma-1} \quad \text{(by (3.3))}$$

so when $\sigma \geqslant 1 + 1/\log \tau$, one has

$$\log(\zeta(s)) < \log\log \tau.$$

It follows ~~that~~ $|\log \zeta(s_1)| \le \log\log \tau$ when

$$s_1 = 1 + \frac{1}{\log \tau} + it.$$

Then, when $1 - \tfrac{1}{2}c/\log \tau \le \sigma \le 1 + 1/\log \tau$,

$$\log \zeta(s) - \log \zeta(s_1) = \int_{s_1}^{s} \frac{\zeta'}{\zeta}(z)\, dz$$

$$\le |s - s_1| \sup_{z \in [s_1, s]} \left| \frac{\zeta'}{\zeta}(z) \right|$$

$$\ll \frac{1}{\log \tau} \cdot \log \tau \ll 1 \quad \text{(using (10.5))}$$

⑥ so $$|\log \zeta(s)| = |\log \zeta(s_1)| + O(1)$$
$$\leq \log\log T + O(1). \quad \square$$

§ 11. The Prime Number Theorem.

Idea: Use Perron's formula to obtain asymptotics for
$$\psi(x) = \sum_{n \leq x} \Lambda(n).$$
via bounds for $\frac{\zeta'}{\zeta}(s)$.

Theorem 11.1. There is a positive constant c having the property ~~that~~
$$\psi(x) = x + O(x \exp(-c\sqrt{\log x})),$$
$$\theta(x) = x + O(x \exp(-c\sqrt{\log x})),$$
and
$$\pi(x) = li(x) + O(x \exp(-c\sqrt{\log x})),$$

⑦ for $x \geq 2$, where

$$li(x) = \int_2^x \frac{dt}{\log t} = x \sum_{k=1}^{K-1} \frac{(k-1)!}{(\log x)^k} + O_K\left(\frac{x}{(\log x)^K}\right)$$

"logarithmic integral"

World Record: Vinogradov - Korobov, 1958:

$$\pi(x) = li(x) + O(x \exp(-c(\log x)^{3/5} (\log\log x)^{-1/5}))$$

Riemann Hypothesis: $\pi(x) = li(x) + O(x^{1/2} \log x)$

$$\pi(x) = li(x) + O(x^{1/2+\varepsilon}) \quad \text{for any } \varepsilon > 0$$

$$\Downarrow$$

RH

$$\Downarrow$$

$$\pi(x) = li(x) + O(x^{1/2} \log x).$$

$$\Omega\left(x^{1/2} \frac{\log\log\log x}{\log x}\right)$$

⑧

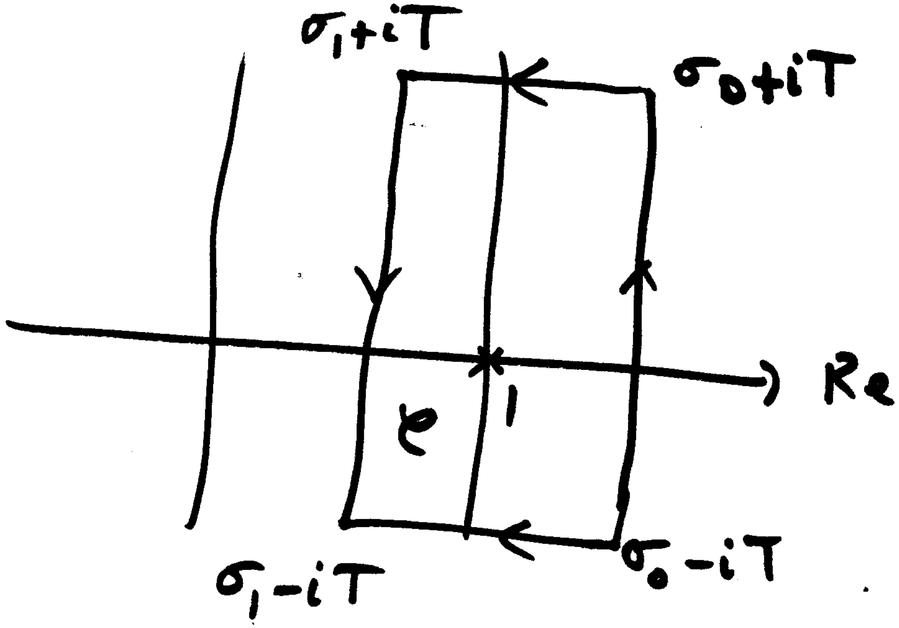
<u>Proof:</u> Recall
$$-\frac{\zeta'}{\zeta}(s) = \sum_{n=1}^{\infty} \Lambda(n) n^{-s}.$$

Apply quantitative version of Perron's formula (Theorem 9.2): for $\sigma_0 > 1$, here
$$\psi(x) = -\frac{1}{2\pi i} \int_{\sigma_0 - iT}^{\sigma_0 + iT} \frac{\zeta'}{\zeta}(s) \frac{x^s}{s} \, ds + R(T),$$

where
$$R(T) \ll \sum_{\frac{x}{2} < n < 2x} \Lambda(n) \min\left\{1, \frac{x}{T|x-n|}\right\} + \left(\frac{(4x)^{\sigma_0}}{T} \sum_{n=1}^{\infty} \frac{\Lambda(n)}{n^{\sigma_0}}\right)$$

Qn 4 of Problem Sheet 4 shows ~~that~~ $2 \leq T \leq x$ and $\sigma_0 = 1 + 1/\log x$, one has $R(T) \ll \frac{x}{T}(\log x)^2$, whence
$$\psi(x) = -\frac{1}{2\pi i} \int_{\sigma_0 - iT}^{\sigma_0 + iT} \frac{\zeta'}{\zeta}(s) \frac{x^s}{s} \, ds + O\left(\frac{x}{T}(\log x)^2\right).$$

9



$$\sigma_1 = 1 - c/\log T$$

21 Oct 2020 **Recall:** **Theorem 11.1** (Prime Number Theorem)

①

There is a positive constant c having the property that

$$\psi(x) = x + O\left(x e^{-c\sqrt{\log x}}\right), \qquad \left(\psi(x) = \sum_{n \le x} \Lambda(n)\right)$$

$$\theta(x) = x + O\left(x e^{-c\sqrt{\log x}}\right),$$

and

$$\pi(x) = li(x) + O\left(x e^{-c\sqrt{\log x}}\right).$$

Note: $$li(x) = \int_2^x \frac{dt}{\log t} = x \sum_{k=1}^{K-1} \frac{(k-1)!}{(\log x)^k} + O_K\left(\frac{x}{(\log x)^K}\right).$$

Proof: Use explicit Perron + Qn 4 of Problem Sheet 4:

When $2 \le T \le x$ and $\sigma_0 = 1 + 1/\log x$, have

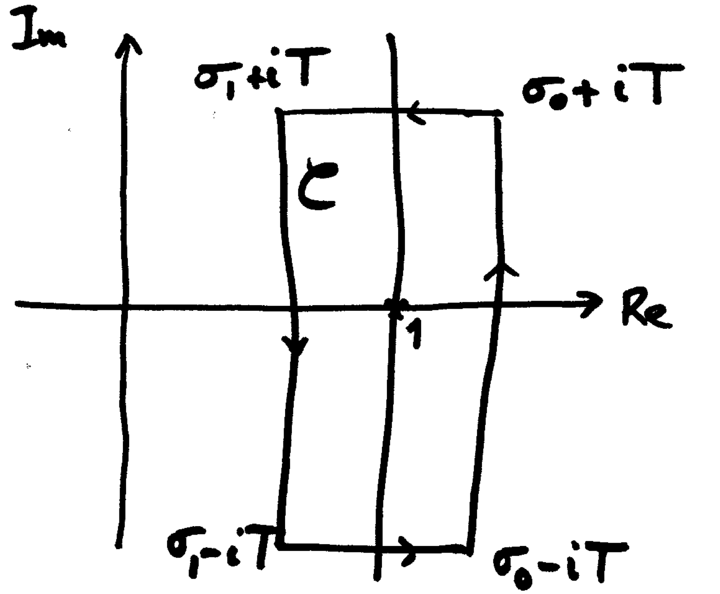
$$\psi(x) = -\frac{1}{2\pi i} \int_{\sigma_0 - iT}^{\sigma_0 + iT} \frac{\zeta'}{\zeta}(s) \frac{x^s}{s}\, ds + O\left(\frac{x}{T} (\log x)^2\right).$$

We may suppose that b is a positive constant with the property that $\zeta(s)$ has no zeros s with $\sigma > 1 - 2b/\log t$. Then $-\frac{\zeta'}{\zeta}(s)$ is analytic within $\mathcal{C}$ except for a simple pole at $s=1$, residue 1.

②

Then

$$\frac{1}{2\pi i}\int_{\mathcal{C}} \left(-\frac{\zeta'}{\zeta}(s)\right)\frac{x^s}{s}\,ds = \lim_{s\to 1}\frac{x^s}{s} = x .$$



$\boxed{\sigma_0 = 1 + 1/\log x \,,\quad \sigma_1 = 1 - b/\log T}$

Also,

$$\int_{\sigma_1 \pm iT}^{\sigma_0 \pm iT} \left(-\frac{\zeta'}{\zeta}(s)\right)\frac{x^s}{s}\,ds \overset{\text{Thm 10.8}}{\ll} \log T \cdot \frac{x^{\sigma_0}}{T}(\sigma_0 - \sigma_1)$$

$(x \geq T)$

$$\ll x/T$$

and

$$\int_{\sigma_1 - iT}^{\sigma_1 + iT} \left(-\frac{\zeta'}{\zeta}(s)\right)\frac{x^s}{s}\,ds \overset{\text{Thm 10.8}}{\ll} \log T \cdot x^{\sigma_1}\int_{-T}^{T}\frac{dt}{1+|t|} \quad \left(|t| \geq \tfrac{7}{8}\right)$$

$$+\, x^{\sigma_1}\int_{-1}^{1}\frac{dt}{|\sigma_1 + it - 1|}$$

$\left(|t| \leq \tfrac{7}{8} \right.$

$\frac{\zeta'}{\zeta}(s) \ll \frac{1}{|s-1|}$

③

$$\ll x^{\sigma_1} \cdot (\log T)^2 + \frac{x^{\sigma_1}}{1-\sigma_1}$$

$$\ll x^{\sigma_1} (\log T)^2.$$

$\frac{\zeta'}{\zeta}(s) \ll \frac{1}{|s-1|}$ for $|t| \le 7/8$

Thus, we conclude ~~that~~

$$\frac{1}{2\pi i} \int_{\sigma_0 - iT}^{\sigma_0 + iT} \left(-\frac{\zeta'}{\zeta}(s) \right) \frac{x^s}{s}\, ds = x + O\left(\frac{x}{T} + x^{\sigma_1} (\log T)^2 \right)$$

$$\Downarrow$$

$$\psi(x) = x + O\left(\frac{x}{T} (\log x)^2 + x^{1 - b/\log T} (\log x)^2 \right).$$

$$= x + O\left(x (\log x)^2 \left(\frac{1}{T} + x^{-b/\log T} \right) \right).$$

Optimise choice of T by arranging that

$$T \asymp x^{b/\log T} \longleftrightarrow \log T \asymp b \frac{\log x}{\log T}$$

i.e. $\log T \approx \sqrt{\log x} \longrightarrow T = \exp(\sqrt{\log x})$.

choose

④ Thus $\psi(x) = x + O\left(x \exp(-c\sqrt{\log x})\right)$,

provided $c < \min\{\tfrac{1}{2}b, 1\}$.

We have $\theta(x) = \psi(x) + O(\sqrt{x})$ (Ex., cf. Problems 3, Qn 1).

and thus $\theta(x) = x + O(x \exp(-c\sqrt{\log x}))$.

Finally, by Riemann-Stieltjes integration,

$$\pi(x) = \int_{2^-}^{x} \frac{1}{\log u}\, d\,\theta(u) = \int_{2^-}^{x} \frac{du}{\log u} + \int_{2^-}^{x} \frac{d(\theta(u)-u)}{\log u}$$

$$= li(x) + \left[\frac{\theta(u)-u}{\log u}\right]_{2^-}^{x} + \int_{2^-}^{x} \frac{\theta(u)-u}{u(\log u)^2}\, du$$

$$\Rightarrow \quad \pi(x) - li(x) \ll x\exp(-c\sqrt{\log x}) + \underbrace{\int_2^x \exp(-c\sqrt{\log u})\, du}_{\ll\, x\exp(-c\sqrt{\log x})}$$

$t = 2^j$ [Hint: $\int_t^{2t} \exp(-c\sqrt{\log u})\, du$]

⑤

Hence
$$\pi(x) = li(x) + O(x \exp(-c\sqrt{\log x})). \qquad \square$$

§ 12. The distribution of smooth numbers.

"smooth number" $\longleftrightarrow$ $n \in \mathbb{N}_{>1}$, all of whose prime factors are "small" ("friable")

R- smooth number n means $p | n$ & p prime ~~then~~ $p \le R$.

Definition 12.1 Let

$$S(x,y) := \{ n \in \mathbb{Z} \cap [1,x] : p | n \text{ \& } p \text{ prime} \Rightarrow p \le y \}$$

$$\psi(x,y) := \operatorname{card}(S(x,y)).$$

Have in mind $y = x^{\eta}$ with $\eta > 0$ small

or

$y = \exp(\sqrt{\log x \log\log x})$

or

$y = (\log x)^c$.

⑥

<u>Dickman function</u> : $\rho(u)$

Defined to be the unique continuous function $\rho : [0, \infty) \to \mathbb{R}$ satisfying differential - delay equation

$$u \rho'(u) = -\rho(u-1) \qquad (\text{for } u > 1)$$

subject to

$$\rho(u) = 1 \quad \text{for} \quad 0 \le u \le 1 .$$

<u>Theorem 12.2</u> (Dickman, 1930) For each $U \ge 0$, one has

$$\psi(x, x^{\frac{1}{u}}) = \rho(u) x + O\left(\frac{x}{\log x}\right) \quad (0 \le u \le U, \; x \ge 2).$$

$x^{\frac{1}{u}}$ - smooth integers

Observe that

$$\rho'(u) = -\frac{\rho(u-1)}{u} \qquad (u > 1),$$

so when $1 \le u \le v$, one has

$$\rho(v) - \rho(u) = \int_u^v \rho'(t)\,dt = -\int_u^v \frac{\rho(t-1)}{t}\,dt. \qquad (12.1)$$

Also, when $u > 1$,

$$(u\rho(u))' = u\rho'(u) + \rho(u) = \rho(u) - \rho(u-1),$$

whence

$$u\rho(u) = \int_{u-1}^{u} \rho(v)\,dv + C, \qquad \text{for } u \geq 1,$$

where

$$C = 1\cdot\rho(1) - \int_0^1 \rho(v)\,dv = 1 - 1 = 0.$$

So (inductively)

$$\rho(u) = \frac{1}{u}\int_{u-1}^{u} \rho(v)\,dv,$$

so by induction $\rho(u) > 0$ for all $u \geq 0$. Also, since

$$\rho'(u) = -\frac{1}{u}\rho(u-1) < 0 \qquad (u > 1),$$

so $\rho(u)$ is decreasing for $u > 1$. By induction

$$\rho(n) \leq \frac{1}{n!} = 1/\Gamma(n+1) \qquad (n \in \mathbb{N}).$$

23 Oct 2020

①

Recall: **§12. The distribution of smooth numbers.**

$n \in \mathbb{N}$ "y-smooth" $\longleftrightarrow$ $p \mid n$ & p prime $\Rightarrow p \le y$.

$S(x,y) := \{ n \le x : n \ y\text{-smooth} \}$

$\psi(x,y) := \operatorname{card} S(x,y)$ "psixyology"

Theorem 12·2. (Dickman, 1930). For each $U \ge 0$, one has

$$\psi(x, x^{1/u}) = \rho(u)\, x + O\left(x/\log x \right) \qquad (0 \le u \le U,\ x \ge 2),$$

where $\rho(u)$ is the Dickman function.

$\rho : [0,\infty) \longrightarrow \mathbb{R}$ is ~~the~~ unique continuous function satisfying

$$u\rho'(u) = -\rho(u-1) \qquad (u > 1)$$

subject to

$$\rho(u) = 1 \qquad (0 \le u < 1).$$

Have

$$\rho(v) - \rho(u) = -\int_u^v \frac{\rho(t-1)}{t}\, dt. \qquad (12·1)$$

$$u\rho(u) = \int_{u-1}^{u} \rho(v)\, dv \Rightarrow \begin{array}{l} \rho(u) > 0 \text{ for } u \ge 0 \\ \rho(u) \text{ decreasing} \end{array}$$

$$\rho'(u) = -\frac{1}{u}\rho(u-1) \quad (u > 1) \Rightarrow \rho(n) \le \frac{1}{n!} = \frac{1}{\Gamma(n+1)} \qquad (n \in \mathbb{N})$$

② <u>Proof of Theorem 12.2.</u> Apply a variant of inclusion-exclusion principle known as Buchstab's identity.

First consider $0 \leq u \leq 1$:

$$\psi(x, x^{1/u}) = \operatorname{card}\{1 \leq n \leq x\} = \lfloor x \rfloor = x + O(1)$$

$$= \underset{(=1)}{\rho(u)}\, x + O(1). \quad \square$$

Next, suppose $1 < u \leq 2$ and put $y = x^{1/u}$. We observe that if $n \leq x$, then n can have at most one prime factor p with $p > x^{1/u} \geq \sqrt{x}$, whence

$$\psi(x,y) = \sum_{1 \leq n \leq x} 1 - \sum_{y < p \leq x} \sum_{\substack{n \leq x \\ p \mid n}} 1$$

$$= \lfloor x \rfloor - \sum_{y < p \leq x} \left\lfloor \frac{x}{p} \right\rfloor$$

$$= x - x \sum_{y < p \leq x} \frac{1}{p} + O(\pi(x))$$

③

$$\overset{\text{Mertens}}{=} x - x\left(\log\log x - \log\log y + O\left(\frac{1}{\log y}\right)\right) + O\left(\frac{x}{\log x}\right)$$

$$= x\left(1 - \log\left(\frac{\log x}{\log y}\right)\right) + O\left(\frac{x}{\log x}\right).$$

But when $1 < u \leq 2$, one has

$$\begin{array}{l} \rho(u) - \rho(1) \overset{(12.1)}{=} -\int_1^u \frac{\rho(t-1)}{t}\,dt = -\int_1^u \frac{dt}{t} = -\log u \\ \quad \| \\ \rho(u) - 1 \qquad \Longrightarrow \qquad \rho(u) = 1 - \log u \quad (1 < u \leq 2) \end{array}$$

Then $$\psi(x,y) = x\rho(u) + O\left(\frac{x}{\log x}\right) \quad \left(u = \frac{\log x}{\log y} = \frac{\log x}{\log(x^{1/u})}\right)$$

We now proceed by induction on U, supposing the conclusion of theorem holds for $0 \leq u \leq U$. This is known to hold for $U = 2$. Seek to establish conclusion for

$$0 \leq u \leq U + 1.$$

④ Suppose when $0 \le u \le U$, one has

$$\psi(x, x^{1/u}) = \rho(u)\, x + O\left(\frac{x}{\log x}\right).$$

Suppose next ~~that~~ $U < u \le U+1$ and put $y = x^{1/u}$.

Observe ~~that~~ when $1 \le n \le x$, and $P(n)$ denotes largest prime factor of n, ~~then~~

$$\psi(x, y) = 1 + \sum_{p \le y} \operatorname{card}\{2 \le n \le x : P(n) = p\}.$$

$$= 1 + \sum_{p \le y} \psi(x/p, p) \qquad \text{(Buchstab)}$$

Hence

$$\psi(x, y) - \psi(x, x^{\frac{1}{U}}) = \sum_{p \le y} \psi(x/p, p) - \sum_{p \le x^{1/U}} \psi(x/p, p)$$

$$= - \sum_{y < p \le x^{1/U}} \psi(x/p, p).$$

Write $u_p = \dfrac{\log(x/p)}{\log p}$, so $\dfrac{x}{p} = p^{u_p}$.

⑤ $(y = x^{1/u})$

Then

$$u_p = \frac{\log x - \log p}{\log p} \le u-1 \le U.$$

Thus,

$$\sum_{y<p\le x^{1/U}} \psi(x/p, p) = \sum_{y<p\le x^{\frac{1}{U}}} \left(\rho\Bigg(\overbrace{\frac{\log x}{\log p} - 1}^{u_p} \Bigg) \frac{x}{p} + O\left(\frac{x/p}{\log(x/p)}\right)\right)$$

(Ind. Hyp.)

$$= x \sum_{y<p\le x^{1/U}} \frac{\rho\left(\frac{\log x}{\log p} - 1\right)}{p} + O\Bigg(\frac{x}{\log x} \underbrace{\sum_{y<p\le x^{\frac{1}{U}}} \frac{1}{p}}_{\text{Mertens}}\Bigg)$$

since

$$\sum_{y<p\le x^{1/U}} \frac{1}{p} = \log\left(\frac{\log x^{1/U}}{\log y}\right) = \log\left(\frac{u}{U}\right) = \overset{O(1)}{\Theta(1)}$$

We estimate main term by Riemann - Stieltjes integration.

Put $A(z) = \sum_{p\le z} \frac{1}{p}$, so $A(z) = \log\log z + c + r(z)$,

where $r(z) = O(1/\log z)$ & $c > 0$ is const.

⑥ Thus

$$\sum_{y<p\le x^{1/U}} \frac{\rho\left(\frac{\log x}{\log p}-1\right)}{p} = \int_y^{x^{1/U}} \rho\left(\frac{\log x}{\log z}-1\right) d\,A(z)$$

$$\overset{R\text{-}S}{=} \int_{x^{1/u}}^{x^{1/U}} \rho\left(\frac{\log x}{\log z}-1\right) d\,(\log\log z) + \int_{x^{1/u}}^{x^{1/U}} \rho\left(\frac{\log x}{\log z}-1\right) dr(z)$$

$$= -\int_u^U \rho(t-1)\frac{dt}{t} + \left[\rho\left(\frac{\log x}{\log z}-1\right) r(z)\right]_{x^{1/u}}^{x^{1/U}} - \int_{x^{1/u}}^{x^{1/U}} r(z)\, d\rho\left(\frac{\log x}{\log z}-1\right)$$

$$t = \frac{\log x}{\log z} \Rightarrow \frac{dt}{t} = \frac{-\log x \cdot dz}{z(\log z)^2 \cdot (\log x)/(\log z)}$$

$$= -d\,(\log\log z) \qquad \boxed{r(z) = O\left(\frac{1}{\log z}\right)}$$

$$\overset{(12.17)}{=} \rho(U)-\rho(u) + O\left(\frac{1}{\log x}\left(1+\int_{x^{1/u}}^{x^{1/U}} d\left|\rho\left(\frac{\log x}{\log z}-1\right)\right|\right)\right)$$

$$= \rho(U)-\rho(u) + O\left(\frac{1}{\log x}\right).$$

7

Thus

$$\sum_{y<p\le x^{1/U}} \psi(x/p, p) = x\left(\rho(U) - \rho(u) + O\left(\frac{1}{\log x}\right)\right),$$

so

$$\psi(x,y) - \psi(x, x^{\frac{1}{U}}) = x\left(\rho(u) - \rho(U) + O\left(\frac{1}{\log x}\right)\right)$$

and since

$$\psi(x, x^{\frac{1}{U}}) = x\rho(U) + O\left(\frac{x}{\log x}\right)$$

$$\Downarrow$$

$$\psi(x,y) = x\rho(u) + O\left(\frac{x}{\log x}\right). \qquad \square$$

This confirms inductive hypothesis for $U \le u \le U+1$, so theorem follows by induction. //

26 Oct 2020

①

Recall: **Theorem 12.2** (Dickman, 1930) For each $U \geq 0$, we have

$$\psi(x, x^{1/u}) = x\rho(u) + O_U\left(\frac{x}{\log x}\right) \qquad (0 \leq u \leq U,\ x \geq 2).$$

~

$\rho : [0, \infty) \longrightarrow \mathbb{R}$ unique continuous function satisfying

$$\left.\begin{array}{ll} u\rho'(u) = -\rho(u-1) & (u > 1) \\ \rho(u) = 1 & (0 \leq u \leq 1) \end{array}\right\}$$

Fact: $\dfrac{1}{2\Gamma(2u+1)} \leq \dfrac{1}{\rho(u)} \leq \dfrac{1}{\Gamma(u+1)}$ $\qquad \rho(u) > 0$, decreasing for $u \geq 0$.

~

Note: Have $\psi(x,y) \gg_\eta x$ when $y \geq x^\eta$, and $\eta > 0$ fixed.

(x^η-smooth numbers in $[1, X]$ have "positive density".)

~

What happens when $y = \exp(c\sqrt{\log x \log\log x})$?

$y = (\log x)^c$? $\qquad (c > 0)$.

② <u>Theorem 12.3.</u> Suppose that $\log x \le y \le x$. Then one has

$$\psi(x,y) \gg \frac{x}{y} \exp\left(-u \log\log x + u/2\right),$$

where $u = \dfrac{\log x}{\log y}$.

[Upper bounds require additional work — we'll punt on this for lack of time, but qualitatively similar to lower bound here].

<u>Corollary 12.4:</u> One has $\psi(x, (\log x)^a) \gg x^{1-\frac{1}{a}+o(1)}$ for $a \ge 1$.

[In fact, also, $\psi(x, (\log x)^a) \ll x^{1-\frac{1}{a}+o(1)}$].

"hypersmooth numbers".

<u>Proof:</u> From Theorem 12.3, one has

$$\psi(x,(\log x)^a) \gg x(\log x)^{-a} \exp\left(-\frac{\log x}{a\log\log x}\cdot \log\log x + \frac{\log x}{2a\log\log x}\right)$$

$$= x^{1-\frac{1}{a}} \exp\Big(\underbrace{\frac{1}{2a}\cdot\frac{\log x}{\log\log x} - a\log\log x}_{=\,o(\log x)}\Big)$$

$$= x^{1-\frac{1}{a}+o(1)}. \qquad \square$$

③

<u>Corollary 12.5.</u> When $c > 0$ is fixed, one has

$$\psi(x, \exp(c\sqrt{\log x \log\log x})) \gg x \exp\left(-\left(c+\frac{1}{c}+o(1)\right)\sqrt{\log x \log\log x}\right).$$

<u>Proof:</u> From Theorem 12.3:

$$\psi(x, \exp(c\sqrt{\log x \log\log x})) \gg x \exp\Big(-c\sqrt{\log x \log\log x} - \frac{\log x}{c\sqrt{\log x \log\log x}} \cdot \log\log x + \frac{1}{2}\frac{\log x}{c\sqrt{\log x \log\log x}}\Big)$$

$$= x \exp\left(-\left(c+\frac{1}{c}\right)\sqrt{\log x \log\log x} + \frac{1}{2c}\sqrt{\frac{\log x}{\log\log x}}\right)$$

$$= x \exp\left(-\left(c+\frac{1}{c}+o(1)\right)\sqrt{\log x \log\log x}\right). \quad //$$

④ <u>Proof.</u> (of Theorem 12.3). (Idea : $n = p_1^{a_1} \cdots p_r^{a_r}$)

Write $r = \pi(y) \sim \frac{y}{\log y}$, and let the first r prime numbers be $p_1, \ldots, p_r$. Then if $n \in S(x,y)$, one has

$$\log n = a_1 \log p_1 + \cdots + a_r \log p_r,$$

for some non-negative integers $a_1, \ldots, a_r$ with $a_i \le \frac{\log n}{\log p_i}$.

Each choice of $\underline{a}$ with $a_1 + \cdots + a_r \le \frac{\log n}{\log p_r}$ gives a unique element $p_1^{a_1} \cdots p_r^{a_r} \in S(x,y)$.

Put

$$k := \left\lfloor \frac{\log n}{\log y} \right\rfloor$$

Thus

$$\begin{aligned}\psi(x,y) &\geq \#\{a_i \in \mathbb{Z}_{\geq 0} : a_1 + \cdots + a_r \leq k\} \\ &= \#\{a_i \in \mathbb{Z}_{\geq 0} : a_0 + a_1 + \cdots + a_r = k\} \\ &= \text{coeff of } t^k \text{ in } \left(\sum_{a=0}^{\infty} t^a\right)^{r+1} = (1-t)^{-r-1}\end{aligned}$$

⑤

$$= \binom{r+k}{k} = \frac{(r+k)!}{r!\,k!} \qquad \left(\begin{aligned} r &= \pi(y) \sim \frac{y}{\log y} \\ k &= \left\lfloor \frac{\log n}{\log y} \right\rfloor \leqslant \frac{\log x}{\log y} =: u \end{aligned} \right)$$

By Stirling's formula, we obtain

$$\psi(x,y) \gg \frac{(r+k)^k}{k^k} \frac{(r+k)^r}{r^r} \cdot \frac{1}{\sqrt{k}} \qquad \text{since } r \ggg k.$$

Put $z = \dfrac{y}{k \log y} = \dfrac{r}{k}(1+o(1))$. Then

$$\psi(x,y) \gg \left(1 + \frac{y}{k\log y}\right)^k \left(1 + \frac{k\log y}{y}\right)^{y/\log y} \cdot \frac{1}{\sqrt{k}}$$

$$\gg \left(z\left(1 + 1/z\right)^z \right)^k$$

Note that

$$z\left(1+\frac{1}{z}\right)^z = \exp\left(z\log\left(1+\frac{1}{z}\right) + \log z \right)$$ is

an increasing function of z for $z \geq 1$, whence

⑥ $$\psi(x,y) \gg \left(W\left(1+\frac{1}{W}\right)^{W}\right)^{k} \geq \left(W\left(1+\frac{1}{W}\right)^{W}\right)^{u-1},$$

where $W = \dfrac{y}{u\log y}$ (since $k \leq u$). But $W \leq \dfrac{y}{\sqrt{k}}$

(since $\log x \leq y \leq x \Rightarrow \dfrac{\log x}{\log\log x} \leq u \leq 1$]), so that

$$\psi(x,y) \gg \frac{1}{y}\cdot\left(\frac{y}{u\log y}\right)^{u}\left(1+\frac{u\log y}{y}\right)^{y/\log y}$$

$$= \frac{1}{y}\cdot e^{u\log y}\exp\left(-u\log\log x + \frac{y}{\log y}\left(\log\left(1+\frac{\log x}{y}\right)\right)\right)$$

$$= \frac{x}{y}\exp\left(-u\log\log x + \frac{1}{2}\underbrace{\frac{\log x}{\log y}}_{= u}\right).$$

<u>Application</u>: Suppose q is large, and ~~that~~ χ is a non-principal character modulo q. Then

$$\sum_{1\leq n\leq q}\chi(n) = 0$$

7

$$\Downarrow$$

$\chi(n) \neq 1$ for some integer n with $(n,q)=1$ & $1 \leq n \leq q$.

Theorem 12·6. When p is a large prime number and χ is non-principal modulo p, there is a prime number π with $\pi \leq p^{1/\sqrt{e}+o(1)}$ for which $\chi(\pi) \neq 1$.

WR: $p^{1/4\sqrt{e}+o(1)}$ Conj: p^{ε} for any $\varepsilon > 0$.

Proof: Suppose $\chi(\pi) = 1$ for all primes π with $1 \leq \pi \leq y$, whence $\chi(m) = 1$ for $m \in S(q,y)$.

Thus

$$0 = \sum_{1 \leq n \leq p-1} \chi(n) = \sum_{m \in S(p,y)} \chi(m) + \sum_{\substack{1 \leq m \leq p-1 \\ m \notin S(p,y)}} \chi(n)$$

$$\geq \psi(p,y) - (p-1-\psi(p,y))$$

So

$$0 \geq 2\psi(p,y) - (p-1)$$

$$\Rightarrow \quad \psi(p,y) \leq \frac{1}{2}(p-1)$$

$$\Rightarrow \quad \frac{\psi(p,y)}{p} \leq \frac{1}{2} + O\left(\frac{1}{p}\right).$$

Then by Theorem 12.2, noting that when $y = p^{1/\sqrt{e}}$, one has

$$\rho\left(\frac{\log p}{\log y}\right) = \rho(\sqrt{e}) = 1 - \log(\sqrt{e}) = \frac{1}{2},$$

Thus the relation $\frac{\psi(p,y)}{p} \leq \frac{1}{2} + O\left(\frac{1}{p}\right)$ ensures that $y \leq p^{\frac{1}{\sqrt{e}} + o(1)}$.

Then necessarily $\chi(\pi) \neq 1$ for some $\pi \leq p^{\frac{1}{\sqrt{e}} + o(1)}$. //

28 Oct 2020

Problems Class on Problem Sheet 3.

①

$$\theta(x;q,a) = \sum_{\substack{p \le x \\ p \equiv a \pmod q}} \log p$$

$$\psi(x;q,a) = \sum_{\substack{n \le x \\ n \equiv a \pmod q}} \Lambda(n) = \sum_{\substack{p^k \le x \\ p^k \equiv a \pmod q \\ k \ge 1}} \log p$$

$$\pi(x;q,a) = \sum_{\substack{p \le x \\ p \equiv a \pmod q}} 1 .$$

Q1] One has $\theta(x;q,a) = \sum_{\substack{p \le x \\ p \equiv a \pmod q}} \log p = \psi(x;q,a) - \sum_{\substack{p^k \le x \\ p^k \equiv a \pmod q \\ k \ge 2}} \log p$,

Whence $|\theta(x;q,a) - \psi(x;q,a)| \le \sum_{p \le x^{1/2}} \left(\frac{\log x}{\log p}\right) \log p = \log x \cdot \pi(x^{\frac{1}{2}}) \ll x^{1/2}.$

② **Q21** R-S integration :

$$\pi(x;q,a) = \int_{2^-}^{x} \frac{1}{\log y}\, d\theta(y;q,a) = \left[\frac{\theta(y;q,a)}{\log y}\right]_{2^-}^{x} + \int_{2^-}^{x} \frac{\theta(y;q,a)}{y\,(\log y)^2}\, dy$$

$$= \frac{\theta(x;q,a)}{\log x} + O\left(\int_{2^-}^{x} \frac{dy}{(\log y)^2}\right), \text{ since } \theta(y;q,a) \le \theta(y) \ll y$$

$$= \frac{\theta(x;q,a)}{\log x} + O\left(\frac{x}{(\log x)^2}\right).$$

~

$$\int_{2^k}^{2^{k+1}} \frac{dy}{(\log y)^2} = \frac{2^k}{k^2}$$

$$\int_{\sqrt{x}}^{x} \frac{dy}{(\log y)^2} \qquad \int_{2^-}^{x} \frac{dy}{(\log y)^2} \sim \sum_{k \le \log_2 x} \frac{2^k}{k^2}$$

③

$\boxed{\text{Q3}}$ $(a,q)=1$:

$$\sum_{\substack{p\le x\\ p\equiv a \,(\mathrm{mod}\, q)}} \frac{\log p}{p} = \frac{1}{\phi(q)}\log x + O_q(1).$$

(i) We may assume ~~that~~, for a suitable $a(q)>0$, giving an upper bound for $O_q(1)$-term, we have

$$\left| \sum_{\substack{p\le x\\ p\equiv a \,(\mathrm{mod}\, q)}} \frac{\log p}{p} - \frac{1}{\phi(q)}\log x \right| \le a(q).$$

Thus, when $C = C(q)$ is large, we deduce ~~that~~

$$\sum_{\substack{x/C<p\le x\\ p\equiv a \,(\mathrm{mod}\, q)}} \frac{\log p}{p} \geq \underbrace{\left(\frac{1}{\phi(q)}\log x - a(q)\right)}_{\uparrow \atop \sum_{\substack{p\le x\\ p\equiv a \,(\mathrm{mod}\, q)}} \frac{\log p}{p}} - \underbrace{\left(\frac{1}{\phi(q)}\log \frac{x}{C} + a(q)\right)}_{\uparrow \atop \sum_{\substack{p\le x/C\\ p\equiv a \,(\mathrm{mod}\, q)}} \frac{\log p}{p}}$$

$$= \frac{1}{\phi(q)}\log C - 2a(q).$$

④ Take $C(q) > \exp(3a(q)\phi(q))$, say, so that

$$\sum_{\substack{x/C < p \leq x \\ p \equiv a \pmod q}} \frac{\log p}{p} \geq \frac{3a(q)\phi(q)}{\phi(q)} - 2a(q) = a(q) > 0.$$

□

(we can take $c(q) = a(q)$.

$$\left[\quad C(q) > \exp\left(2a(q)\phi(q) + \cancel{\phi(q)}\right). \right]$$

(ii) $$\underbrace{\sum_{\substack{x/C < p \leq x \\ p \equiv a \pmod q}} \frac{\log p}{p}}_{\substack{\vee \\ c(q)}} \leq \sum_{\substack{x/C < p \leq x \\ p \equiv a \pmod q}} \frac{\log x}{x/C} \leq \frac{C\log x}{x} \sum_{\substack{p \leq x \\ p \equiv a \pmod q}} 1$$

$$\Rightarrow \underbrace{\sum_{\substack{p \leq x \\ p \equiv a \pmod q}} 1}_{\pi(x;q,a)} \geq \frac{c}{C}\frac{x}{\log x} \gg_q \frac{x}{\log x}.$$ □

⑤ <u>Q4</u> (i)

$$\frac{1}{\phi(q)} \sum_{\chi \in X(q)} \Big| \sum_{\substack{n=1 \\ (n,q)=1}}^{q} a_n \chi(n) \Big|^2 = \cancel{\frac{1}{\phi(q)} \sum_{\chi \in X(q)} \sum_{(n,q)=1}}$$

$\chi(n) = 0$ if $(n,q) > 1$.

$$= \frac{1}{\phi(q)} \sum_{\chi \in X(q)} \sum_{(n,q)=1} \sum_{(m,q)=1} a_n \chi(n) \cdot \overline{a_m}\, \overline{\chi}(m)$$

$\overline{\chi}(m) = \chi(m^{-1})$

$$= \frac{1}{\phi(q)} \sum_{(n,q)=1} \sum_{(m,q)=1} a_n \overline{a_m} \underbrace{\sum_{\chi \in X(q)} \chi(nm^{-1})}_{\begin{cases} = 0 & \text{if } nm^{-1} \not\equiv 1 \ (q) \\ = \phi(q) & \text{if } nm^{-1} \equiv 1 \ (q) \end{cases}}$$

$$= \sum_{\substack{1 \le n \le q \\ (n,q)=1}} |a_n|^2 . \qquad \square$$

(ii)

$$\frac{1}{\phi(q)} \sum_{\substack{n=1\\(n,q)=1}}^{q} \Big| \sum_{\chi\in X(q)} a_\chi\, \chi(n) \Big|^2$$

$$= \frac{1}{\phi(q)} \sum_{\substack{n=1\\(n,q)=1}}^{q} \sum_{\chi_1\in X(q)} \sum_{\chi_2\in X(q)} a_{\chi_1}\, \underbrace{\chi_1(n)\, \overline{a_{\chi_2}}\, \overline{\chi_2}(n)}_{\chi_1\bar{\chi}_2(n) \qquad \chi_1\bar{\chi}_2\in X(q)}$$

$$= \frac{1}{\phi(q)} \sum_{\chi_1,\chi_2\in X(q)} a_{\chi_1}\, \overline{a_{\chi_2}} \sum_{\substack{n=1\\(n,q)=1}}^{q} (\chi_1\bar{\chi}_2)(n)$$

$$\sum_{\substack{n=1\\(n,q)=1}}^{q} (\chi_1\bar{\chi}_2)(n) = \begin{cases} 0, & \text{when } \chi_1\bar{\chi}_2\neq\chi_0 \\ \phi(q), & \text{when } \chi_1\bar{\chi}_2=\chi_0 \end{cases}$$

$$= \sum_{\chi\in X(q)} |a_\chi|^2 .$$

□

Gauss sums $\quad \sum_{x=1}^{q} \chi(x)\, e(x/q) = \tau(\chi).$

30 Oct 2020

1

§13. The functional equation for $\zeta(s)$.

Recall that $\zeta(s) := \sum_{n=1}^{\infty} n^{-s}$ for $\sigma > 1$

and defined by analytic continuation for $\sigma > 0$ (analytic except for a simple pole with residue 1 at $s = 1$).

Question: Can we analytically continue to $\mathbb{C} \setminus \{1\}$?

Theorem 13.1 The function

$$\xi(s) = \tfrac{1}{2} s(s-1)\zeta(s)\Gamma(\tfrac{1}{2}s)\pi^{-s/2}$$

is entire, and one ha

$$\xi(s) = \xi(1-s) \quad \text{for all } s \in \mathbb{C}$$ (the functional equation).

Proof exploits Poisson summation via "reciprocity" of theta functions. Proceed in some generality for later use.

②

Theorem 13.2. Let $\alpha \in \mathbb{R}$ and $z \in \mathbb{C}$ satisfy $\mathrm{Re}(z) > 0$. Then one has

$$\sum_{n=-\infty}^{\infty} e^{-\pi(n+\alpha)^2 z} = z^{-1/2} \sum_{k=-\infty}^{\infty} e(k\alpha)\, e^{-\pi k^2/z}. \qquad \text{——— (13.1)}$$

Moreover, under the same conditions,

$$\sum_{n=-\infty}^{\infty} (n+\alpha)\, e^{-\pi(n+\alpha)^2 z} = -iz^{-3/2} \sum_{k=-\infty}^{\infty} k\, e(k\alpha)\, e^{-\pi k^2/z}. \qquad \text{— (13.2)}$$

Here, the branch of $z^{1/2}$ is determined by taking $1^{1/2} = 1$.

Proof: Study lhs of (13.1), writing $f(u) = e^{-\pi u^2 z}$. Notice that lhs of (13.1) is $\sum_{n \in \mathbb{Z}} f(n+\alpha)$, and when $\mathrm{Re}(z) > 0$, this sum is absolutely convergent for each α, and uniformly convergent for α in any compact set. The Fourier transform of f is given by

③ $$\hat{f}(t) := \int_{\mathbb{R}} f(u)\, e(-tu)\, du,$$

so that if $g(u) = f(u+\alpha)$, we have

$$\hat{g}(t) = \int_{\mathbb{R}} f(u+\alpha)\, e(-tu)\, du = \int_{\mathbb{R}} f(v)\, e(-t(v-\alpha))\, dv$$

$$= e(t\alpha)\, \hat{f}(t).$$

Moreover,

$$\hat{f}(t) = \int_{-\infty}^{\infty} \exp\left(-\pi u^2 z - 2\pi i t u\right) du$$

$$\left(u^2 z + 2itu = (u + it/z)^2 z + t^2/z \right)$$

$$= e^{-\pi t^2/z} \int_{-\infty}^{\infty} e^{-\pi (u + it/z)^2 z}\, du$$

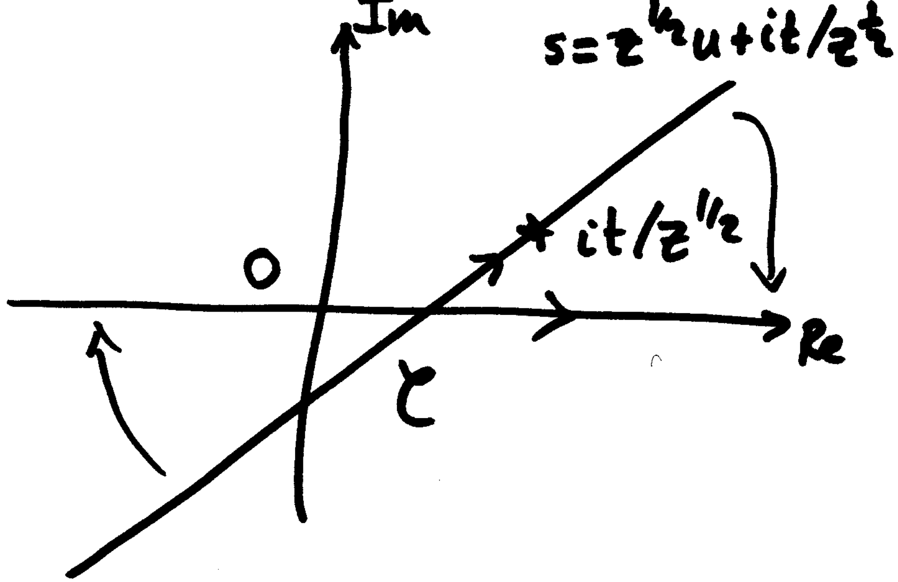
$$s := z^{1/2} u + it/z^{\frac{1}{2}}$$

$$= e^{-\pi t^2/z} \int_{-\infty}^{\infty} e^{-\pi (z^{1/2} u + it/z^{1/2})^2}\, du$$

$$\hat{f}(t) = z^{-\frac{1}{2}} \cdot e^{-\pi t^2/z} \int_{C} e^{-\pi s^2}\, ds$$

④

We can move the contour $\mathcal{C}$ defined by $s = z^{1/2}u + it/z^{\frac{1}{2}}$ to the real line, since when $|u|$ is large, say $|u| > R$, we have



$$|e^{-\pi s^2}| = e^{-\pi u^2 \operatorname{Re}(z)} = O\left(e^{-\pi R^2 \cdot \operatorname{Re}(z)}\right)$$

Thus, we have
$$\hat{f}(t) = z^{-\frac{1}{2}} e^{-\pi t^2/z} \int_{-\infty}^{\infty} e^{-\pi s^2}\, ds$$
$$= z^{-\frac{1}{2}} e^{-\pi t^2/z}.$$

We may conclude thus far ~~that~~

$$\sum_{k\in\mathbb{Z}} \hat{g}(k) = \sum_{k\in\mathbb{Z}} e(k\alpha)\hat{f}(k) = \sum_{k\in\mathbb{Z}} e(k\alpha)\cdot z^{-\frac{1}{2}} e^{-\pi k^2/z}.$$

(5) Since $\mathrm{Re}(z) > 0$, the Fourier transforms $\hat{g}(k) = e(k\alpha)\hat{f}(k)$ are rapidly decaying as $|k| \to \infty$, so by Poisson summation we obtain

$$\sum_{k\in\mathbb{Z}} \hat{g}(k) = \sum_{n\in\mathbb{Z}} g(n) = \sum_{n\in\mathbb{Z}} f(n+\alpha) = \sum_{n\in\mathbb{Z}} e^{-\pi(n+\alpha)^2 z}.$$

Thus

$$\sum_{n\in\mathbb{Z}} e^{-\pi(n+\alpha)^2 z} = z^{-\frac{1}{2}} \sum_{k\in\mathbb{Z}} e(k\alpha)\, e^{-\pi k^2/z}. \quad \square \quad (18.4)$$

To obtain (13.2), ~~differentiate~~ wrt α noting the uniform convergence of the series for α in compact sets:

$$\frac{d}{d\alpha}\left(\sum_{n=-\infty}^{\infty} \exp\left(-\pi(n+\alpha)^2 z\right)\right) = \frac{d}{d\alpha}\left(z^{-\frac{1}{2}} \sum_{k\in\mathbb{Z}} e(k\alpha)\, e^{-\pi k^2/z}\right)$$

$$\sum_{n=-\infty}^{\infty} -2\pi(n+\alpha) z \exp\left(-\pi(n+\alpha)^2 z\right) \qquad z^{-\frac{1}{2}} \sum_{k=-\infty}^{\infty} 2\pi i k\, e(k\alpha)\, e^{-\pi k^2/z}$$

(6)

$$\Downarrow$$

$$\sum_{n\in\mathbb{Z}} (n+\alpha)\, e^{-\pi(n+\alpha)^2 z} = -i\, z^{-\frac{3}{2}} \sum_{k\in\mathbb{Z}} k\, e(k\alpha)\, e^{-\pi k^2/z}.$$

$\square$ (13.2)

Define <u>incomplete</u> gamma function (for $a\in\mathbb{C}$) by

$$\Gamma(s,a) = \int_a^\infty e^{-w}\, w^{s-1}\, dw \qquad (\Gamma(s)=\Gamma(s,0))$$

<u>Theorem 13.3</u> Suppose that $s\in\mathbb{C}\setminus\{0,1\}$ and $\mathrm{Re}(z)\geq 0$. Then one has

$$\zeta(s)\,\Gamma(s/2)\,\pi^{-s/2} = \pi^{-s/2}\sum_{n=1}^{\infty}\Gamma(s/2, \pi n^2 z)\, n^{-s}$$

$$+\,\pi^{(s-1)/2}\sum_{n=1}^{\infty}\Gamma((1-s)/2, \pi n^2/z)\, n^{s-1}$$

$$+\,\frac{z^{(s-1)/2}}{s-1} - \frac{z^{s/2}}{s}.$$

7

<u>Proof of Theorem 13.1</u> (assuming Thm 13.3). Assume rhs terms in Thm 13.3 are all analytic functions of s. for $s \in \mathbb{C} \setminus \{0,1\}$. We may observe that the rhs of Thm 13.3 is left unchanged with the interchange

$$s \longleftrightarrow 1-s$$

$$z \longleftrightarrow 1/z$$

Thus the lhs is also left unchanged by same operations, and we have

$$\zeta(s)\,\Gamma(s/2)\,\pi^{-s/2} = \zeta(1-s)\,\Gamma\left(\tfrac{1-s}{2}\right)\pi^{-(1-s)/2},$$

except possibly when $s \in \{0,1\}$. If we multiply by $s(1-s)$, then the pole of $\zeta(s)$ at $s=1$ is cancelled, and identity of Thm 13.1 holds when $s \in \{0,1\}$. $\square$. Note also lhs is analytic.

02 Nov 2020 §13. The functional equation for $\zeta(s)$.

① Recall: goal is to prove: Theorem 13·1 The function

$$\xi(s) = \tfrac{1}{2}s(s-1)\,\zeta(s)\,\Gamma(\tfrac{1}{2}s)\,\pi^{-\frac{1}{2}s}$$

is entire, and one has $\xi(s) = \xi(1-s)$ $(s \in \mathbb{C})$.

Theorem 13·2. Let $\alpha \in \mathbb{R}$ and $z \in \mathbb{C}$ satisfy $\mathrm{Re}(z) > 0$. Then one has

$$\sum_{n=-\infty}^{\infty} e^{-\pi(n+\alpha)^2 z} = z^{-\frac{1}{2}} \sum_{k=-\infty}^{\infty} e(k\alpha)\, e^{-\pi k^2/z}, \qquad (13\cdot1)$$

$$\sum_{n=-\infty}^{\infty} (n+\alpha)\, e^{-\pi(n+\alpha)^2 z} = -i z^{-3/2} \sum_{k=-\infty}^{\infty} k e(k\alpha)\, e^{-\pi k^2/z}. \qquad (13\cdot2)$$

[$1^{1/2} = 1$ determines branch of $z^{1/2}$].

Incomplete gamma function: $\Gamma(s,a) = \int_a^\infty e^{-w} w^{s-1}\,dw$ $(a \in \mathbb{C})$.

Theorem 13·3. Suppose that $s \in \mathbb{C} \setminus \{0,1\}$ and $\mathrm{Re}(z) \geq 0$. Then

$$\zeta(s)\Gamma(s/2)\pi^{-s/2} = \pi^{-s/2}\sum_{n=1}^{\infty}\Gamma(s/2, \pi n^2 z)\, n^{-s} + \underbrace{\pi^{(s-1)/2}}\sum_{n=1}^{\infty}\Gamma((1-s)/2, \pi n^2/z)\, n^{s-1} + z^{(s-1)/2}/(s-1) - z^{s/2}/s.$$

② <u>Recall</u>: rhs invariant under $s \longleftrightarrow 1-s$

$z \longleftrightarrow 1/z$

<u>Proof</u>: Recall that when $\sigma > 0$, one has

$$\Gamma(s) = \int_0^\infty x^{s-1} e^{-x}\, dx .$$

Then

$$\Gamma(s/2) = \int_0^\infty e^{-x} x^{s/2 - 1}\, dx \underset{x = \pi n^2 u}{=} n^s \pi^{\frac{s}{2}} \underbrace{\int_0^\infty e^{-\pi n^2 u} u^{\frac{s}{2}-1}\, du}. \quad (*)$$

Thus, when $\sigma > 1$ (required to obtain abs. conv. of $\sum n^{-s}$), we have

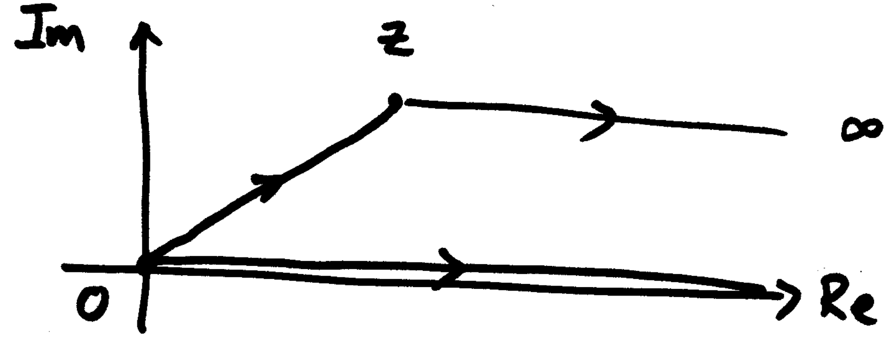
$$\sum_{n=1}^\infty \int_0^\infty e^{-\pi n^2 u} u^{\frac{s}{2}-1}\, du = \int_0^\infty u^{s/2 - 1} \left(\sum_{n=1}^\infty e^{-\pi n^2 u} \right) du$$

$$\| \; (*)$$

$$\sum_{n=1}^\infty n^{-s} \pi^{-s/2} \Gamma(s/2) = \zeta(s) \pi^{-s/2} \Gamma(s/2) .$$

We next adjust integral on rhs to rewrite in terms of incomplete Γ-functions:

③ When $\mathrm{Re}(z) \geq 0$, we move the path $[0,\infty)$ of integ^n to $[0,z] \cup [z,\infty)$. Then



writing
$$\theta_1(u) = \sum_{n=1}^{\infty} e^{-\pi n^2 u}, \quad \text{we see that}$$

$$\int_0^{\infty} \theta_1(u)\, u^{s/2-1}\, du = \underbrace{\int_0^{z} \theta_1(u)\, u^{s/2-1}\, du}_{I_1} + \underbrace{\int_z^{\infty} \theta_1(u)\, u^{s/2-1}\, du}_{I_2}.$$

Reverse course with I_2:

$$I_2 = \sum_{n=1}^{\infty} \int_z^{\infty} e^{-\pi n^2 u}\, u^{s/2-1}\, du \qquad u = x/(n^2\pi)$$

$$= \sum_{n=1}^{\infty} n^{-s} \pi^{-s/2} \int_{\pi n^2 z}^{\infty} e^{-x} x^{s/2-1}\, dx$$

$$= \pi^{-s/2} \sum_{n=1}^{\infty} n^{-s}\, \Gamma(s/2, \pi n^2 z).$$

④ Thus far, we have shown ~~that~~

$$\zeta(s)\,\pi^{-s/2}\,\Gamma(s/2) = \pi^{-s/2}\sum_{n=1}^{\infty} n^{-s}\,\Gamma(s/2, \pi n^2 z) + I_1 .$$

To tackle I_1, put $\theta(u) = 1 + 2\theta_1(u) = \sum_{n\in\mathbb{Z}} e^{-\pi n^2 u}$.

Then

$$I_1 = \int_0^z \overbrace{\tfrac{1}{2}(\theta(u)-1)}^{\theta_1(u)}\, u^{s/2-1}\,du$$

$$= \frac{1}{2}\int_0^z \theta(u)\,u^{s/2-1}\,du - \underbrace{\frac{1}{2}\int_0^z u^{s/2-1}\,du}_{\frac{1}{s} z^{s/2}} .$$

But by Theorem 13.2 $(\alpha=0)$, we have $\theta(u) = u^{-\frac{1}{2}}\theta(1/u)$.

Thus

$$I_1 = \underbrace{\frac{1}{2}\int_0^z \theta\left(\frac{1}{u}\right) u^{(s-3)/2}\,du}_{\substack{\shortparallel \quad v = 1/u \\ \int_{\frac{1}{z}}^{\infty} \theta(v)\, v^{-(s+1)/2}\,dv}} - \frac{1}{s} z^{s/2}$$

5

Then $$I_1 = \int_{1/z}^{\infty} \theta_1(v)\, v^{-(s+1)/2}\, dv + \frac{1}{2}\int_{1/z}^{\infty} v^{-(s+1)/2}\, dv$$

$$-1/s \; z^{s/2}$$

$$= \int_{\frac{1}{z}}^{\infty} \theta_1(v)\, v^{-(s+1)/2}\, dv + \frac{1}{s-1} z^{(s-1)/2} - \frac{1}{s} z^{s/2}.$$

Then $$I_1 + \frac{1}{s} z^{s/2} - \frac{1}{s-1} z^{(s-1)/2} = \sum_{n=1}^{\infty} \int_{1/z}^{\infty} e^{-\pi n^2 v}\, v^{-(s+1)/2}\, dv$$

$$v = x/(n^2\pi)$$

$$= \sum_{n=1}^{\infty} n^{s-1}\, \pi^{(s-1)/2} \int_{\pi n^2/z}^{\infty} e^{-x}\, x^{-(s+1)/2}\, dx$$

$$= \pi^{(s-1)/2} \sum_{n=1}^{\infty} n^{s-1}\, \Gamma\left(\frac{1-s}{2}, \pi n^2/z\right).$$

Then $$\zeta(s)\pi^{-s/2}\Gamma(s/2) = \pi^{-s/2}\sum_{n=1}^{\infty} n^{-s}\, \Gamma(s/2, \pi n^2 z) + \pi^{(s-1)/2}\sum_{n=1}^{\infty} n^{s-1}\, \Gamma\left(\frac{1-s}{2}, \pi n^2/z\right)$$
$$- \frac{1}{s} z^{s/2} + \frac{1}{s-1} z^{(s-1)/2}. \qquad \square$$

⑥ This establishes Theorem 13·3 for $Re(s) > 1$. However, the last two terms on rhs are analytic for $s \in \mathbb{C} \setminus \{0,1\}$. The first two terms are also analytic for $s \in \mathbb{C} \setminus \{0,1\}$, as we now show. Observe that for each z with $Re(z) > 0$ and $|z| \geqslant \varepsilon > 0$, one has

$$\Gamma(s, z) = \int_z^\infty e^{-u} u^{s-1}\, du$$

$$= \int_0^\infty e^{-v-z} (z+v)^{s-1}\, dv$$

$|z+v|^{s-1} \ll |z|^{\sigma-1}$ uniformly

$$\ll |z|^{\sigma-1}.$$

for $|\sigma| \leq C$, fixed $C>0$.

$$\Rightarrow \quad n^{-s}\,\Gamma(s/2, \pi n^2 z) \ll_\varepsilon n^{-\sigma} (n^2)^{\sigma/2 - 1} = n^{-2}.$$

Thus $\sum_{n=1}^{\infty} n^{-s}\,\Gamma(s/2, \pi n^2 z)$ is uniformly convergent

⑦ for s in any compact set, hence (Weierstrass) is entire. Likewise for

$$\sum_{n=1}^{\infty} n^{s-1}\,\Gamma\left((1-s)/2,\ \pi n^2/z\right)$$

since

$$n^{s-1}\,\Gamma\left(\frac{1-s}{2},\ \pi n^2 z\right) \ll_\varepsilon n^{\sigma-1}\cdot (n^2)^{\frac{1-\sigma}{2}-1} \ll n^{-2}.$$

Then the four terms on rhs in the statement of Theorem 13.3 are analytic, except possibly at $s=0,1$.

By the uniqueness of analytic continuation, therefore, the identity claimed in Theorem 13.3 holds for all $s \in \mathbb{C}\setminus\{0,1\}$. //

4 Nov 2020 | Q11 ①

Perron's formula

$$\psi(x) = \sum_{n\le x} \Lambda(n), \text{ and } \sum_{n=1}^{\infty} \frac{\Lambda(n)}{n^s} = -\frac{\zeta'}{\zeta}(s) \quad Re(s)>1.$$

$$\boxed{\sum_{n\le x}{}^{①} \Lambda(n) = \lim_{T\to\infty} \frac{1}{2\pi i}\int_{\sigma_0-iT}^{\sigma_0+iT} -\frac{\zeta'}{\zeta}(s)\frac{x^s}{s}\,ds \qquad x>0,\ x\notin\mathbb{Z} \quad (\sigma_0>1)}$$

$$\sum_{n=1}^{\infty} \frac{\mu(n)}{n^s} = \prod_p \sum_{h=0}^{\infty} \frac{\mu(p^h)}{p^{hs}} = \prod_p\left(1-\frac{1}{p^s}\right) \quad Re(s)>1$$

$$= 1/\zeta(s) \qquad \left(\zeta(s) = \prod_p \left(1-\frac{1}{p^s}\right)^{-1}\right)$$

⇓ Perron.

$$\sum_{n\le x}{}' \mu(n) = \lim_{T\to\infty} \frac{1}{2\pi i}\int_{\sigma_0-iT}^{\sigma_0+iT} \frac{1}{\zeta(s)}\frac{x^s}{s}\,ds$$

$$\left[\ll x\exp(-c\sqrt{\log x}) \text{ for } c>0\right].$$

②

$$\sum_{n=1}^{\infty} \frac{\mu(n)^2}{n^s} = \prod_p \left(\sum_{h=0}^{\infty} \frac{\mu(p^h)^2}{p^{hs}} \right) = \prod_p \left(1 + \frac{1}{p^s} \right)$$

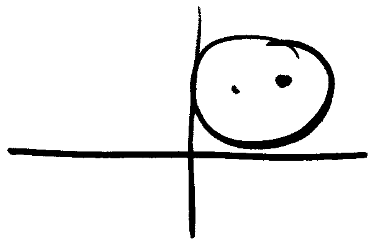
$$= \prod_p \left(\frac{1 - p^{-2s}}{1 - p^{-s}} \right) = \frac{\zeta(s)}{\zeta(2s)} \qquad Re(s) > 1$$

Perron $\Rightarrow$

$$\sum_{1 \le n \le x} \mu(n)^2 = \lim_{T \to \infty} \frac{1}{2\pi i} \int_{\sigma_0 - iT}^{\sigma_0 + iT} \frac{\zeta(s)}{\zeta(2s)} \frac{x^s}{s} ds.$$

Q2 Can assume $\left| \frac{h^{(k)}(0)}{k!} \right| \le \frac{2M}{R^k}$. $(k \ge 1)$

$|z| \le r < R$. — use Taylor:



$$h^{(m)}(z) = \sum_{k=0}^{\infty} \frac{h^{(k+m)}(0)}{k!} z^k$$

$$\Rightarrow \quad |h^{(m)}(z)| \le \sum_{k=0}^{\infty} \left| \frac{h^{(k+m)}(0)}{k!} \right| r^k \le \frac{2M}{R^m} \sum_{k=0}^{\infty} \frac{(k+m)!}{k! R^k} r^k$$

③

$$\leq \frac{2M}{R^m}\left(\frac{d^m}{dz^m}\frac{1}{1-z}\right)_{z=r/R}$$

$$= \frac{2M}{R^m}\frac{m!}{(1-r/R)^{m+1}}$$

$$\frac{d^m}{dz^m}\left(1+z+\cdots+z^{k+m}+\cdots\right)$$

$$= \frac{2MR}{(R-r)^{m+1}}. \quad (m \geq 1).$$

Q3 Apply Lemma 10·3 to $g(z) := (z-z_0)f(z)$.

$f(z)$ has a simple pole at $z=z_0$, $g(0) = -z_0 f(0) \neq 0$.

$$-\frac{g'}{g}(z) = \sum_{k=1}^{n}\frac{1}{z-z_k} + O\left(\log\left(\frac{M}{|z_0 f(0)|}\right)\right)$$

$$\parallel$$

$$-\left(\frac{f'}{f}(z) + \frac{1}{z-z_0}\right)$$

$|(z-z_0)f(z)| \leq M$

④

<u>Q3 (ii)</u> Apply part (i) to $\zeta(s)$ with pole at $s=1$.

Have
$$\underbrace{(s-1)\zeta(s)}_{g(s)} = (s-1)\left(\frac{1}{s-1} + C_0 + \cdots\right)$$
$$= 1 + C_0(s-1) + \cdots$$
$$\ll 1 \quad \text{for} \quad |s-1| \le 1.$$

Have $(s-1)\zeta(s) \ll t^2$ for t large.

Also have $f(1) \neq 1$. So from (i) gives

$$-\frac{\zeta'}{\zeta}(s) = \frac{1}{s-1} - \sum_\rho \frac{1}{s-\rho} + O\left(\log\left|\frac{t^2}{1}\right|\right)$$

Lemma 10.5 : When $|t| \geq 7/8$ & $5/6 \le \sigma \le 2$ have
$$\frac{\zeta'}{\zeta}(s) = \sum_\rho \frac{1}{s-\rho} + O(\log(|t|+4)).$$

But $\frac{1}{s-1} \ll 1$

$$\Rightarrow \zeta'/\zeta(s) = -\frac{1}{s-1} + \sum_\rho \frac{1}{s-\rho} + O(\log(|t|+4))$$

⑤ When $|t| \le 7/8$ & $5/6 \le \sigma \le 2$, then we can apply part (i) (if necessary in two patches)

$$-\frac{\zeta'}{\zeta}(s) = \frac{1}{s-1} - \sum_{\rho} \frac{1}{s-\rho} + O\left(\log(|t|+4)\right).$$

⑥ Q4 (i) $\zeta(s) = \frac{s}{s-1} - s\int_1^\infty \frac{\{u\}}{u^{s+1}}\,ds = \frac{1}{s-1} + O(1)$ as $s \to 1$,

$$\Rightarrow \quad \frac{\zeta'}{\zeta} = \left(\frac{-\frac{1}{(s-1)^2} - \int_1^\infty \frac{\{u\}}{u^{s+1}}\,ds + s(s+1)\int_1^\infty \frac{\{u\}}{u^{s+2}}\,ds}{\frac{1}{s-1} + O(1)} \right)$$

$$= -\frac{1}{s-1} + O(1).$$

$\Rightarrow \quad -\frac{\zeta'}{\zeta}(\sigma) = \frac{1}{\sigma-1} + O(1)$ for ~~[illegible]~~ $\sigma - 1$ small. $\square$

Hence

$$\frac{4^\sigma + X^\sigma}{T} \sum_{n=1}^\infty \frac{\Lambda(n)}{n^\sigma} = \frac{4^\sigma + X^\sigma}{T}\left(-\frac{\zeta'}{\zeta}(\sigma)\right) \ll \frac{4^\sigma + X^\sigma}{T} \cdot \frac{1}{\sigma-1}$$

$$\ll \frac{(4X)^\sigma}{T(\sigma-1)}.$$

⑦ (ii)

$$\sum_{x/2<n<2x} \underset{\substack{\uparrow \\ \Lambda(n) \ll \log n.}}{\Lambda(n)} \min\left\{1, \frac{x}{T|x-n|}\right\} \ll (\log x)\Big(\underset{\substack{\uparrow \\ |n-x|\le\frac{1}{2}}}{1} + \frac{x}{T}\sum_{k=1}^{x}\frac{1}{k/2}\Big)$$

$$\ll (\log x)\cdot\Big(1 + \frac{x}{T}\underset{\substack{\uparrow \\ \text{upper bound for } \sum_{k\le x}\frac{1}{k}.}}{\log x}\Big)$$

(iii) Explicit Perron:

$$\sum_{n\le x}{}' \Lambda(n) = \psi(x) + O(\log x)$$

$$= \frac{1}{2\pi i}\int_{\sigma_0-iT}^{\sigma_0+iT}\left(-\frac{\zeta'}{\zeta}\right)(s)\cdot\frac{x^s}{s}\,ds + R(T),$$

where

$$R(T) \ll \frac{4^{\sigma_0}+x^{\sigma_0}}{T}\sum_{n=1}^{\infty}\frac{\Lambda(n)}{n^{\sigma_0}} + \sum_{\frac{x}{2}<n\le 2x}\Lambda(n)\min\left\{1, \frac{x}{T|x-n|}\right\}$$

$$\ll \frac{(4x)^{\sigma_0}}{T(\sigma_0-1)} + \frac{x}{T}(\log x)^2$$

(8)

But $\quad \sigma_0 = 1 + 1/\log x$,

so $\quad x^{\sigma_0} = x \cdot \exp((\sigma_0 - 1)\log x) \ll x$

So $\quad R(T) \ll \frac{x}{T}(\log x)^2$. $\square$.//

06 Nov 2020

①

Recall: The functional equation of $\zeta(s)$.

$\zeta(s)$ is the (unique) function defined by $\zeta(s) = \sum_{n=1}^{\infty} n^{-s}$ for $\mathrm{Re}(s) > 1$, and by analytic continuation for $\mathrm{Re}(s) \leq 1$ (simple pole at $s = 1$)

Theorem 13.1: The function

$$\xi(s) = \tfrac{1}{2} s(s-1)\zeta(s)\Gamma(\tfrac{1}{2}s)\pi^{-s/2}$$

is entire, and one has $\xi(s) = \xi(1-s)$ for all $s \in \mathbb{C}$. (functional equation).

Corollary 13.4. When $s \neq 1$, one has

$$\zeta(s) = \zeta(1-s) \cdot 2^{s} \pi^{s-1} \Gamma(1-s) \sin(\tfrac{1}{2}\pi s).$$

Proof. (Using standard properties of gamma function).

$$\Gamma(s)\Gamma(1-s) = \frac{\pi}{\sin(\pi s)} \quad \text{and} \quad \Gamma(s)\Gamma(s+\tfrac{1}{2}) = \sqrt{\pi}\, 2^{1-2s}\Gamma(2s)$$

(2) We have

$$\overset{\xi(s)}{\overset{\parallel}{\tfrac{1}{2}s(s-1)\zeta(s)\Gamma(\tfrac{1}{2}s)\pi^{-s/2}}} = \overset{\xi(1-s)}{\overset{\parallel}{\tfrac{1}{2}(1-s)(-s)\zeta(1-s)\Gamma(\tfrac{1}{2}(1-s))\pi^{\frac{s-1}{2}}}}$$

$$\Downarrow$$

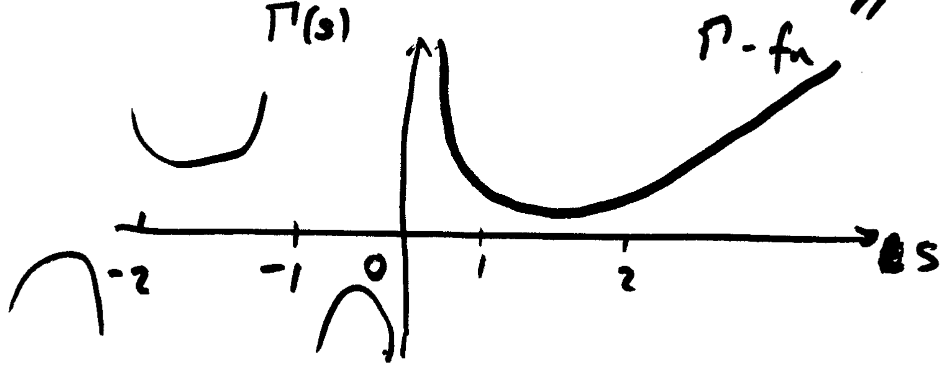
$$\zeta(s) = \zeta(1-s)\,\frac{\Gamma(\frac{1}{2}(1-s))}{\Gamma(\frac{1}{2}s)}\,\pi^{s-\frac{1}{2}}.$$

Have $\displaystyle\frac{1}{\Gamma(\frac{1}{2}s)} = \frac{\sin(\pi s/2)}{\pi}\cdot\Gamma(1-\tfrac{1}{2}s)$

$$\Rightarrow \frac{\Gamma(\frac{1}{2}-\frac{1}{2}s)}{\Gamma(\frac{1}{2}s)} = \frac{\sin(\frac{1}{2}\pi s)}{\pi}\Gamma(\tfrac{1}{2}-\tfrac{1}{2}s)\Gamma(1-\tfrac{1}{2}s)$$

$$= \frac{\sin(\frac{1}{2}\pi s)}{\pi}\sqrt{\pi}\,2^{s}\,\Gamma(1-s)$$

So $\zeta(s) = \zeta(1-s)\,2^{s}\,\pi^{s-1}\sin(\tfrac{1}{2}\pi s)\,\Gamma(1-s).$ //



③ Put $s = -\sigma$, with $\sigma \overset{\mathbb{R}}{>} 0$, we see that

$$\zeta(-\sigma) = \zeta(\sigma+1)\, 2^{-\sigma} \pi^{-\sigma-1} \Gamma(\sigma+1) \sin\left(-\tfrac{1}{2}\pi\sigma\right).$$

We know $\zeta(\sigma+1) \neq 0$ for $\sigma > 0$, and $\zeta(s) \neq 0$ for $\mathrm{Re}(s) > 0$, and thus

$$\zeta(-\sigma) = 0 \quad \text{if and only if} \quad \sin\left(\tfrac{1}{2}\pi\sigma\right) = 0.$$

$$\updownarrow$$

$$\sigma \in 2\mathbb{Z}$$

<u>Corollary</u> <u>13.5.</u> Other than the <u>trivial</u> zeros $-2, -4, \ldots$, all zeros ρ of $\zeta(s)$ satisfy

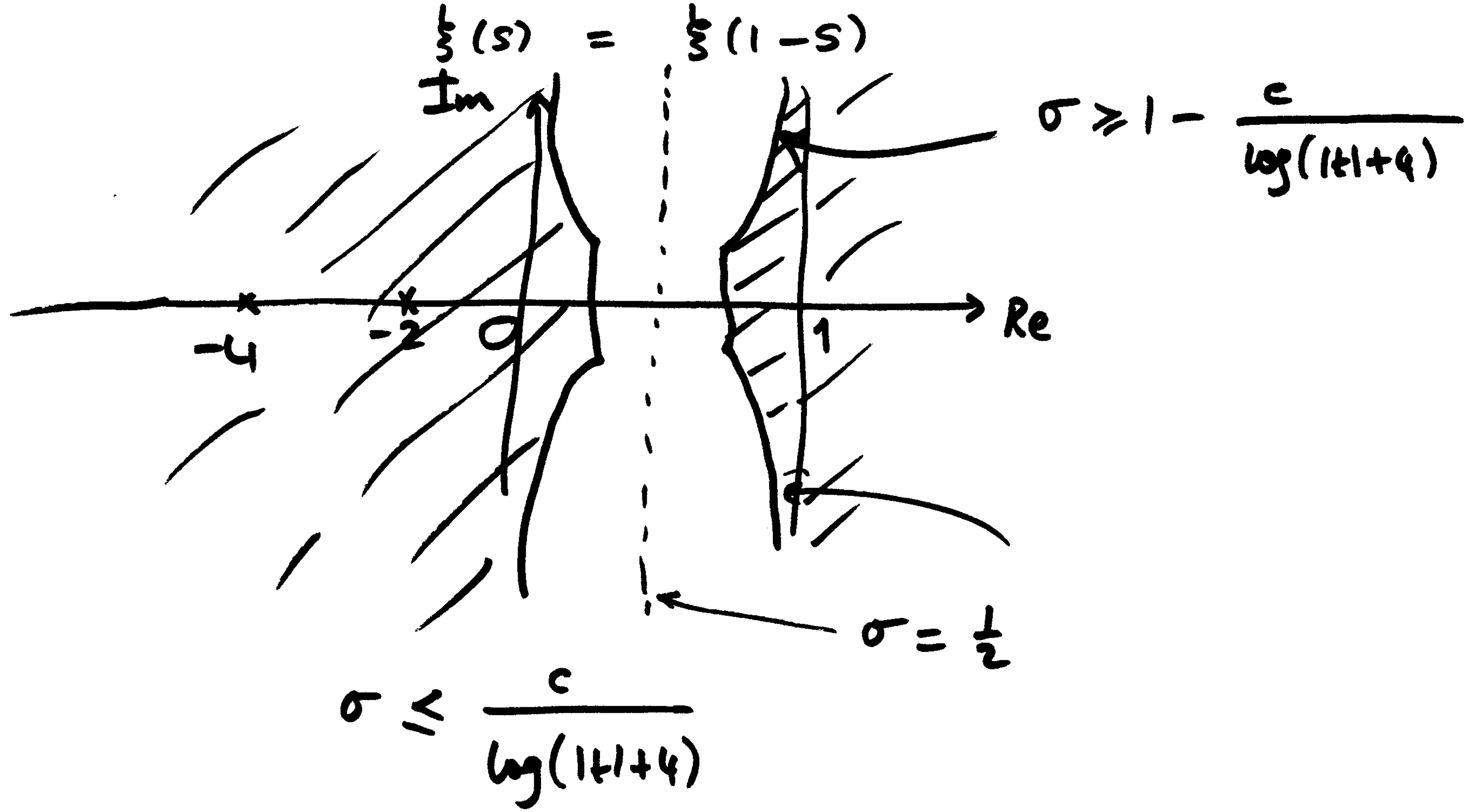
$$0 < \mathrm{Re}(\rho) < 1.$$

<u>Eg:</u>

$$\zeta(0) = \lim_{\sigma \to 0} \zeta(1-\sigma)\, 2^{\sigma} \pi^{\sigma-1} \Gamma(1-\sigma) \sin\left(\tfrac{1}{2}\pi\sigma\right)$$

$$= \underset{\times \Gamma(1)}{\pi^{-1}} \cdot \tfrac{1}{2}\pi \lim_{\sigma \to 0} \sigma \cdot \left(\frac{-1}{\sigma}\right) = -\frac{1}{2}.$$

4



<u>Riemann - Hypothesis :</u> All non-trivial zeros of $\zeta(s)$ lie on $Re(s) = \frac{1}{2}$.

5

§14. A zero-free region for Dirichlet L-functions.

Dirichlet's thm on primes in arith. prog.

$a \in \mathbb{Z}$, $q \in \mathbb{N}$ with $(a,q)=1$

$$\sum_{\substack{p \le x \\ p \equiv a \ (\mathrm{mod}\, q)}} \frac{\log p}{p} = \frac{1}{\phi(q)} \log x + O_q(1)$$

?

$$\pi(x;q,a) := \sum_{\substack{p \le x \\ p \equiv a \ (\mathrm{mod}\, q)}} 1 = \frac{1}{\phi(q)} \mathrm{li}(x) + \text{error}$$

$x/\log x \sim \mathrm{li}(x)$

$$L(s,\chi) = \sum_{n=1}^{\infty} \frac{\chi(n)}{n^s}$$

χ Dirichlet character modulo q.

Seek an analogue of Lemma 10.5 — entails bounding $L(s,\chi)$ via analogue of Lemma 10.4.

⑥ <u>Lemma 14.1.</u> Let χ be a non-principal character modulo q, and suppose ~~that~~ $\delta > 0$ is fixed. Then one has

$$L(s,\chi) \ll \left(1 + (q\tau)^{1-\sigma}\right) \min\left\{\frac{1}{|\sigma - 1|}, \log(q\tau)\right\},$$

uniformly for $\delta \leq \sigma \leq 2$. $(\tau = |t| + 4)$.

<u>Proof:</u> Write

$$S(u,\chi) = \sum_{1 \leq n \leq u} \chi(n).$$

Then by Riemann-Stieltjes integration, one finds ~~that~~

$$L(s,\chi) = \sum_{n \leq x} \chi(n) n^{-s} + \int_{x^-}^{\infty} u^{-s}\, d\,S(u,\chi)$$

$$= \sum_{n \leq x} \chi(n) n^{-s} + \left[S(u,\chi) u^{-s}\right]_{x^-}^{\infty} - \int_{x}^{\infty} S(u,\chi)\, du^{-s}.$$

(7)

$$L(s,\chi) = \sum_{n\le x} \chi(n) n^{-s} - S(x^-,\chi)x^{-s} + s\int_x^\infty u^{-s-1} S(u,\chi)\,du. \qquad Re(s)>0$$

By orthogonality $\sum_{n=m+1}^{m+q} \chi(n) = 0$ for all m, whence

$$|S(u,\chi)| \le q.$$

We take $x = q\tau$, and then last 2 terms contribute

$$\ll q\cdot(q\tau)^{-\sigma} + q\tau \left| \int_x^\infty u^{-\sigma-1}\,du \right| \ll (q\tau)^{1-\sigma}.$$

In the proof of Lemma 10.4 we showed that

$$\sum_{n\le x} |n^{-s}| \ll (1+x^{1-\sigma}) \min\left\{\frac{1}{|1-\sigma|}, \log x\right\},$$

whence

$$\sum_{n\le x} |\chi(n) n^{-s}| \ll (1+x^{1-\sigma}) \min\left\{\frac{1}{|1-\sigma|}, \log x\right\}$$

$$\ll (1+(q\tau)^{1-\sigma}) \min\left\{\frac{1}{|\sigma-1|}, \log(q\tau)\right\}$$

$$\Rightarrow \quad L(s,\chi) \ll (1+(q\tau)^{1-\sigma}) \min\left\{\frac{1}{|\sigma-1|}, \log(q\tau)\right\}. \quad //$$

(8)

Lemma 14.2

Analogue of Lemma 10.5. With $E_0(\chi)=\begin{cases}1, & \chi=\chi_0\\ 0, & \chi\neq\chi_0\end{cases}$

Suppose that χ is a character modulo q. Then whenever $5/6 \le \sigma \le 2$, one has

$$-\frac{L'}{L}(s,\chi) = \frac{E_0(\chi)}{s-1} - \sum_{\rho} \frac{1}{s-\rho} + O(\log(q\tau)),$$

where the sum is over all zeros ρ of $L(s,\chi)$ with

$$\left|\rho - \left(\tfrac{3}{2}+it\right)\right| \le \frac{5}{6}.$$

9 Nov 2020 Recall: Zero-free regions for $L(s,\chi)$.

1

Lemma 14·1. Let χ be a non-principal character modulo q, and suppose that $\delta > 0$ is fixed. Then one has

$$L(s,\chi) \ll (1 + (q\tau)^{1-\sigma}) \min\left\{\frac{1}{|\sigma - 1|}, \log(q\tau)\right\},$$

uniformly for $\delta \le \sigma \le 2$.

$$E_0(\chi) := \begin{cases} 1, & \text{when } \chi = \chi_0, \\ 0, & \text{when } \chi \ne \chi_0. \end{cases}$$

Analogue of Lemma 10·5:

Lemma 14·2. Suppose that χ is a character modulo q. Then whenever $5/6 \le \sigma \le 2$, one has

$$-\frac{L'}{L}(s,\chi) = \frac{E_0(\chi)}{s-1} - \sum \frac{1}{s-\rho} + O(\log(q\tau)),$$

where the sum is over all zeros ρ of $L(s,\chi)$ with $|\rho - (\frac{3}{2} + it)| \le \frac{5}{6}$.

Proof: Suppose first $\chi \ne \chi_0$, so $E_0(\chi) = 0$. We apply Lemma 10·3 with $f(z) = L(z + (\frac{3}{2} + it), \chi)$, $R = 5/6$

② Lemma 10·3: • f analytic in $|z| \le 1$ } $0 < r < R < 1$ &
• $|f(z)| \le M$ and $f(0) \neq 0$ } $|z| \le r$

$$\longrightarrow \quad \frac{f'}{f}(z) = \sum_{k=1}^{n} \frac{1}{z - z_k} + O_{r,R}\left(\log\left(\frac{M}{|f(0)|}\right)\right). \qquad |z| \le R.$$

zeros $z_1, \dots, z_k$

---

and $r = 2/3$. From Lemma 14·1, when $|z| \le 1$, then

$$f(z) \ll (1 + (q\tau)^{1/2}) \log(q\tau), \text{ for } Re(z) \le \tfrac{1}{2},$$

and

$$f(z) = L(z + (\tfrac{3}{2} + it), \chi) \ll \sum_{n=1}^{\infty} n^{-2} \ll 1, \text{ for } Re(z) > \tfrac{1}{2}.$$

Thus $f(z) \ll (q\tau)^{1/2} \log(q\tau) \ll q\tau$ and we can take $M = O(q\tau)$ in application of Lemma 10·3.

Also,

$$|f(0)| = |L(\tfrac{3}{2} + it, \chi)| = \prod_p (1 - \chi(p) p^{-\frac{3}{2} - it})^{-1}$$

$$\ge \prod_p (1 + p^{-3/2})^{-1} \gg 1.$$

Then

$$-\frac{L'}{L}(s, \chi) = - \sum_\rho \frac{1}{s - \rho} + O\Big(\underbrace{\log\left(\frac{M}{|f(0)|}\right)}_{= O(\log(q\tau))}\Big),$$

②

where the summation is over zeros ρ of $L(s,\chi)$ with $\left|\rho - \left(\frac{3}{2}+it\right)\right| \leq \frac{5}{6}$. $\square$

Suppose next that $\chi = \chi_0$. Then we have

$$(*) \qquad L(s,\chi_0) = \zeta(s) \prod_{p\mid q} (1-p^{-s}) \qquad \text{for } \sigma > 1.$$

<u>Note:</u> The rhs is analytic for $s \in \mathbb{C}$ except for a simple pole at $s=1$. The function $L(s,\chi_0)$ is therefore given by rhs for $s \in \mathbb{C}\setminus\{1\}$ by uniqueness of analytic continuation. The zeros of $L(s,\chi_0)$ are given by the zeros of $\zeta(s)$ together with the zeros of $1-p^{-s}$ for $p\mid q$. The latter are given by

$$s = \frac{2\pi i\, k}{\log p} \qquad (k \in \mathbb{Z}).$$

All such zeros satisfy

$$\left| \frac{2\pi i}{\log p} k - \left(\frac{3}{2}+it\right)\right| \geq \frac{3}{2},$$

④ so are excluded from sum over ρ in ~~statement~~ of lemma. Thus

$$\frac{L'}{L}(s, \chi_0) = \frac{\frac{d}{ds}\left(\zeta(s) \prod_{p \mid q} (1-p^{-s}) \right)}{\zeta(s) \prod_{p \mid q} (1-p^{-s})}$$

$$= \frac{\zeta'}{\zeta}(s) + \sum_{p \mid q} \frac{\log p}{p^s - 1}.$$

Here, when $\sigma \geq 5/6$, we have

$$\sum_{p \mid q} \frac{\log p}{p^s - 1} \ll \sum_{p \mid q} \frac{\log p}{p^{\sigma}} \ll \omega(q) \ll \log q.$$

Making use of Lemma 10.5 to ~~estimate~~ $\frac{\zeta'}{\zeta}(s)$ (or we can use Problem set 4, Qn 3(ii)),

$$-\frac{L'}{L}(s, \chi_0) = \frac{1}{s-1} - \sum_{\rho} \frac{1}{s-\rho} + O(\log \tau) + O(\log q)$$

$$= \frac{E_0(\chi_0)}{s-1} - \sum_{\rho} \frac{1}{s-\rho} + O(\log(q\tau)) \quad \square //$$

⑤

<u>Lemma 14.3.</u> When $\sigma > 1$, one has

$$\mathrm{Re}\left(-3\frac{L'}{L}(\sigma, \chi_0) - 4\frac{L'}{L}(\sigma+it, \chi) - \frac{L'}{L}(\sigma+2it, \chi^2)\right) \geq 0.$$

<u>Proof:</u> The lhs here is

$$\mathrm{Re}\left(\sum_{\substack{n=1 \\ (n,q)=1}}^{\infty} \frac{\Lambda(n)}{n^\sigma} \left(3 + 4\chi(n) n^{-it} + \chi(n)^2 n^{-2it}\right)\right). \quad \text{—(14.1)}$$

Thus, since when $(n,q)=1$, one has

$$\chi(n)\, n^{-it} = \exp\left(-it\log n + 2\pi i \xi_n\right),$$

for a suitable $\xi_n \in [0,1)$. Thus, for some $\theta_n \in \mathbb{R}$, we have $\chi(n)\, n^{-it} = e^{i\theta_n}$, whence (14.1) becomes

$$\sum_{\substack{n=1 \\ (n,q)=1}}^{\infty} \frac{\Lambda(n)}{n^\sigma} \Big(\underbrace{3 + 4\cos\theta_n + \cos(2\theta_n)}_{=\,2(1+\cos\theta_n)^2 \,\geq\, 0} \Big).$$

//

⑥ <u>Theorem 14·4</u> There is an absolute constant $c > 0$ with the following property:

(i) If χ is <u>not</u> a quadratic character modulo q, then $L(s,\chi)$ has <u>no</u> zeros in the region

$$\sigma > 1 - \frac{c}{\log(q\tau)} ;$$

(ii) If χ is a quadratic character modulo q, then $L(s,\chi)$ has at most one zero in the region

$$\sigma > 1 - \frac{c}{\log(q\tau)} ;$$

If such a zero β exists, then it is necessarily real and satisfies $\beta < 1$.

["Siegel zero" β].

$$\left[\ \underset{\chi \neq \chi_0}{\chi^2 = \chi_0} \longleftarrow \text{quadratic characters} \right]$$

(7) Proof: When $\chi = \chi_0$, we have

$$L(s,\chi_0) = \zeta(s) \prod_{p|q} (1-p^{-s}),$$

so that zeros of $L(s,\chi_0)$ are those of $\zeta(s)$ together with zeros of $1-p^{-s}$ $(p|q)$. There are no zeros of the former with $\sigma > 1 - c_1/\log \tau$, for suitable $c_1 > 0$. The factors $1-p^{-s}$ have zeros precisely where

$$\exp(s \log p) = 1 \iff s = \frac{2k\pi i}{\log p} \quad (k \in \mathbb{Z}),$$

and again, none of these zeros satisfy $\sigma > 1 - c/\log(q\tau)$.

We suppose henceforth that $\chi \neq \chi_0$. Here, observe first that in view of Euler product

$$L(s,\chi) = \prod_p \left(1 - \frac{\chi(p)}{p^s}\right)^{-1} \quad (\sigma > 1),$$

8

We have
$$\mathrm{Re}\,(L(s,\chi)) \geqslant \prod_p (1+p^{-\sigma})^{-1} > 0 \quad (\sigma > 1).$$

Thus $L(s,\chi) \neq 0$ for $\sigma > 1$. We may now focus on $\sigma \leq 1$.

We suppose, by way of deriving a contradiction, that $L(s,\chi)$ has a zero $\rho_0 = \beta_0 + i\gamma_0$ with $\gamma_0 \in \mathbb{R}$ and $\frac{12}{13} \leq \beta_0 \leq 1$.

- χ complex
- χ quadratic

11 Nov 2020

①

Recall: **Lemma 14·2.** Suppose that χ is a character modulo q. Then whenever $5/6 \le \sigma \le 2$, one has

$$\boxed{E_0(\chi) = \begin{cases} 1, & \chi = \chi_0 \\ 0, & \chi \ne \chi_0 \end{cases}}$$

$$-\frac{L'}{L}(s,\chi) = \frac{E_0(\chi)}{s-1} - \sum_{\rho} \frac{1}{s-\rho} + O(\log(q\tau)),$$

where the sum is over all zeros ρ of $L(s,\chi)$, with $|\rho - (\frac{3}{2} + it)| \le \frac{5}{6}$.

Lemma 14·3. When $\sigma > 1$, one has

$$\mathrm{Re}\left(-3\frac{L'}{L}(\sigma,\chi_0) - 4\frac{L'}{L}(\sigma+it,\chi) - \frac{L'}{L}(\sigma+2it,\chi^2)\right) \ge 0.$$

Theorem 14·4. There is an absolute constant $c>0$ with the following property:

(i) If χ is <u>not</u> a quadratic character modulo q, then $L(s,\chi)$ has no zeros in the region $\sigma > 1 - c/\log(q\tau)$;

(ii) If χ is a quadratic character modulo q, then $L(s,\chi)$ has at most one zero in the region $\sigma > 1 - c/\log(q\tau)$.

2 If such a zero β exists, then it is necessarily real and satisfies $\beta < 1$.

<u>Proof:</u> • If $\chi = \chi_0$ (so χ real but not quadratic) then
$$L(s,\chi_0) = \zeta(s) \prod_{p \mid q} (1 - p^{-s}),$$
so follows from theory of $\zeta(s)$.

• No zeros of $L(s,\chi) = \prod_p \left(1 - \frac{\chi(p)}{p^s}\right)^{-1}$ $(\chi \neq \chi_0)$
when $\sigma > 1$.

Seek a contradiction by supposing $L(s,\chi)$ has a zero $\rho_0 = \beta_0 + i\gamma_0$ with $\gamma_0 \in \mathbb{R}$ and $\frac{12}{13} \leq \beta_0 \leq 1$.

Divide into cases: $(L(1,\chi) \neq 0)$

<u>(i) Suppose χ complex.</u> Thus $\chi(n)$ is <u>not</u> real-valued.

We seek to apply Lemma 14·3:
$$\mathrm{Re}\left(-3\frac{L'}{L}(\sigma,\chi_0) - 4\frac{L'}{L}(\sigma+it,\chi) - \frac{L'}{L}(\sigma+2it,\chi^2)\right) \geq 0,$$

③ and apply argument of Theorem 10.7. Thus, we consider a small $\delta > 0$ and note that when $s = \sigma + it$ and $\sigma = 1+\delta$, then for all zeros ρ of $L(s,\chi)$ we have

$$\mathrm{Re}(s-\rho) > 0 \quad \text{and} \quad \mathrm{Re}\left(\frac{1}{s-\rho}\right) > 0.$$

Hence, by Lemma 14.2, for a suitable const. $c_1 > 0$, we have

$$-\mathrm{Re}\left(\frac{L'}{L}(\overset{\sigma}{\widetilde{1+\delta}}, \chi_0)\right) = \mathrm{Re}\left(\frac{1}{\delta} - \sum_{\substack{\rho \\ (*)\,|\rho - (\frac{3}{2}+it)| \leq \frac{5}{6}}}^{*} \frac{1}{1+\delta-\rho}\right) + c_1 \log(4q)$$

$$\leq \frac{1}{\delta} + c_1 \log(4q),$$

$$-\mathrm{Re}\left(\frac{L'}{L}(1+\delta+i\gamma_0, \chi)\right) = \mathrm{Re}\left(-\sum_{\rho}^{*} \frac{1}{1+\delta+i\gamma_0-\rho}\right) + c_1 \log(q(|\gamma_0|+4))$$

$$(\rho_0 = \beta_0 + i\gamma_0)$$

$$\leq -\frac{1}{1+\delta-\beta_0} + c_1 \log(q(|\gamma_0|+4)).$$

and

$$-\mathrm{Re}\left(\frac{L'}{L}(1+\delta+2i\gamma_0, \underset{\neq \chi_0}{\chi^2})\right) \leq \overbrace{\cancel{\mathrm{Re}\left(-\sum_{\rho'}^{*} \frac{1}{1+\delta+2i\gamma_0-\rho'}\right)}}^{\leq 0} + c_1 \log(q(|\gamma_0|+4)) \qquad (14.2)$$

0

ρ' zeros of $L(s,\chi^2)$

④ Take linear combination to obtain:

$$0 \leq \frac{3}{\delta} - \frac{4}{1+\delta-\beta_0} + \underbrace{(3+4+1)}_{8} c_1 \log(q(|\gamma_0|+4)).$$

If one were to have $\beta_0 = 1$, then

$$-\frac{1}{\delta} + 8c_1 \log(q(|\gamma_0|+4)) \geq 0 \quad \# \text{ for } \delta \text{ suff small.}$$

Thus $\beta_0 < 1$. If we take $\delta = 6(1-\beta_0)$, whence

$$\frac{3}{6(1-\beta_0)} - \frac{4}{7(1-\beta_0)} + 8c_1 \log(q(|\gamma_0|+4)) \geq 0$$

$$\Updownarrow$$

$$1-\beta_0 \geq \frac{1}{14 \cdot 8c_1 \log(q(|\gamma_0|+4))}.$$

Thus $L(s,\chi)$ has no zeros in region $\sigma > 1 - \frac{c}{\log(q\tau)}$, provided $c < 1/112c_1$. $\square$

(ii) Suppose ~~that~~ χ is quadratic and $|\gamma_0| \geq 6(1-\beta_0)$.

⑤ Know that $L(1,\chi)\neq 0$ (from proof of Dirichlet's thm).

Thus if $\gamma_0 = 0$, then our <u>hypothesis</u> implies $1-\beta_0 \le \frac{1}{6}|\gamma_0| = 0$, so $\beta_0 \geq 1$ ※ since knows $\beta_0 \le 1$ and $L(1,\chi)\neq 0$ (so $\beta_0 \neq 1$).

Thus we have $\gamma_0 \neq 0$ and also $\chi^2 = \chi_0$. So now replace (14.2) by

$$-\mathrm{Re}\left(\frac{L'}{L}(1+\delta+2i\gamma_0, \underset{\chi_0}{\underset{\|}{\chi^2}})\right) \le \mathrm{Re}\left(\frac{1}{\delta+2i\gamma_0}\right) + \underset{\text{cont}^{\text{ns}}\text{ of } \rho'}{\underset{\uparrow}{0}} + c_1 \log(q(|\gamma_0|+4))$$

$$= \frac{\delta}{\delta^2+4\gamma_0^2} + c_1 \log(q(|\gamma_0|+4))$$

Now Lemma 14.3 gives

$$0 \le \frac{3}{\delta} - \frac{4}{1+\delta-\beta_0} + \frac{\delta}{\delta^2+4\gamma_0^2} + 8c_1 \log(q(|\gamma_0|+4)).$$

If $\beta_0 = 1$ then letting $\delta \to 0+$ we obtain contradiction.

So can suppose $\beta_0 < 1$. Since $|\gamma_0| \geq 6(1-\beta_0)$, ⑥ on taking $\delta = 6(1-\beta_0)$, we obtain

$$0 \leq \frac{1}{1-\beta_0}\underbrace{\left(\frac{3}{6} - \frac{4}{7} + \frac{6}{6^2+4\cdot 6^2}\right)} + 8c_1 \log(q(|\gamma_0|+4))$$

$$\Rightarrow \quad \frac{1}{1-\beta_0}\left(-\frac{1}{2} + \frac{4}{7} - \frac{1}{30}\right) \leq 8c_1 \log(q(|\gamma_0|+4))$$

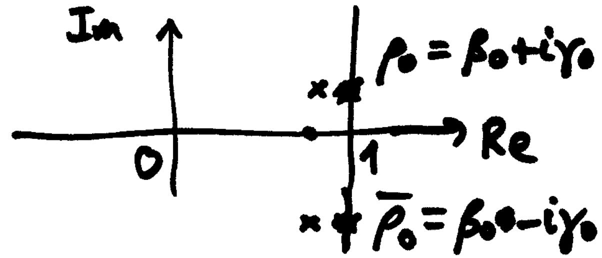
$$\Updownarrow$$

$$1-\beta_0 \geqslant \frac{1}{\frac{105}{4}\cdot 8c_1 \log(q(|\gamma_0|+4))} = \frac{1}{210c_1 \log(q|\gamma_0|+4))}$$

Then $L(s,\chi)$ has no zeros in $\sigma > 1 - c/\log(q\tau)$ provided $c < 1/210c_1$, subject to $|\gamma_0| \geqslant 6(1-\beta_0)$. □

<u>(iii) Suppose that χ is quadratic and $0 < |\gamma_0| \leq 6(1-\beta_0)$.</u>

$\rho_0 = \beta_0 + i\gamma_0$ not real!



13 Nov 2020

①

Recall: Theorem 14·4. There is an absolute constant $c > 0$ with the following property:

(i) If χ is not a quadratic character modulo q, then $L(s, \chi)$ has no zeros in the region $\sigma > 1 - c/\log(q\tau)$;

(ii) If χ is a quadratic character modulo q, then $L(s, \chi)$ has at most one zero in the region

$$\sigma > 1 - c/\log(q\tau).$$

If such a zero β exists, then it is real and $\beta < 1$.

Proof: • $L(s, \chi_0)$ ✓ $\qquad \rho_0 = \beta_0 + i\gamma_0$ a zero with $\frac{12}{13} \leq \beta \leq 1$

(i) χ complex ✓ $\qquad$ (ii) χ quadratic and $|\gamma_0| \geq 6(1-\beta_0)$ ✓

(iii) Suppose that χ is quadratic and $0 < |\gamma_0| \leq 6(1-\beta_0)$.

Suppose there is a non-real zero $\rho_0 = \beta_0 + i\gamma_0$ in the putative zero-free region with $|\gamma_0| \leq 6(1-\beta_0)$. By conjugation (Schwarz reflection principle) there is a second zero $\beta_0 - i\gamma_0$ $\left(= \overline{\rho_0}\right)$.

(note: $L(\beta_0 - i\gamma_0, \chi) = \overline{L(\beta_0 + i\gamma_0, \chi)} = 0$). $\quad (\overline{\chi} = \chi)$

②

In Lemma 14·2, when $1 < \sigma \le 2$, one has

$$\mathrm{Re}\left(-\frac{L'}{L}(\sigma,\chi)\right) \le -\mathrm{Re}\left(\frac{1}{\sigma-\rho_0}+\frac{1}{\sigma-\overline{\rho_0}}\right)+c_1\log(4q),$$

for a suitable constant $c_1>0$. Hence $\boxed{\dfrac{1}{\sigma-\beta_0-i\gamma_0}+\dfrac{1}{\sigma-\beta_0+i\gamma_0}}$

$$\mathrm{Re}\left(-\frac{L'}{L}(\sigma,\chi)\right) \le -\frac{2(\sigma-\beta_0)}{(\sigma-\beta_0)^2+\gamma_0^2}+c_1\log(4q). \quad \text{— (14·5)}$$

Note in this instance, when $\sigma>1$, one

$$-\frac{L'}{L}(\sigma,\chi_0)-\frac{L'}{L}(\sigma,\chi)=\sum_{\substack{n=1\\(n,q)=1}}^{\infty}\frac{\Lambda(n)}{n^{\sigma}}\left(1+\chi(n)\right)\ge 0.$$

Put $\sigma = 1+\overbrace{13(1-\beta_0)}^{\delta}$ and recall ~~that~~

$$-\mathrm{Re}\left(\frac{L'}{L}(1+\delta,\chi_0)\right)\le\frac{1}{\delta}+c_1\log q.$$

Thus,

$$0\le\frac{1}{\delta}-\frac{2(\sigma-\beta_0)}{(\sigma-\beta_0)^2+\gamma_0^2}+2c_1\log(4q),$$

③ Whence

$$\frac{1}{1-\beta_0}\left(\frac{1}{13}-\underbrace{\frac{2\,(13+1)}{(13+1)^2+6^2}}_{\frac{7}{7^2+3^2}}\right)+2c_1\log(4q)\geq 0$$

Thus $$1-\beta_0 \geq \frac{1}{\frac{754}{33}\cdot 2c_1\log(4q)}.$$

Then $L(s,\chi)$ has no zeros in region $\sigma > 1 - {}^{c}/_{\log(4q)}$, provided that $c < {}^{1}/_{46c_1}$, subject to $0<|\gamma_0|\leq 6(1-\beta_0)$. $\square$

<u>(iv) Suppose that χ is quadratic and $\gamma_0=0$: so ρ_0 is real.</u>

If β_0 is a real zero of $L(s,\chi)$, then $\beta_0<1$ as a consequence of our proof ~~that~~ $L(1,\chi)\neq 0$. Suppose that β_0 and β_1 are two such real zeros, say with $\beta_0\leq\beta_1<1$. Then by Lemma 14·2, ~~that~~ when

④ $1 \leq \sigma \leq 2$, one has

$$\mathrm{Re}\left(-\frac{L'}{L}(\sigma,\chi)\right) \leq -\frac{1}{\sigma-\beta_0} - \frac{1}{\sigma-\beta_1} + c_1 \log(4q)$$

$$\leq -\frac{2}{\sigma-\beta_0} + c_1 \log(4q).$$

As in case (iii), we have

$$-\frac{L'}{L}(\sigma,\chi_0) - \frac{L'}{L}(\sigma,\chi) \geq 0,$$

whence with $\sigma = 1+\delta$, we have

$$0 \leq \frac{1}{\delta} - \frac{2}{\sigma-\beta_0} + 2c_1 \log(4q).$$

Put $\delta = 2(1-\beta_0)$. Then we obtain,

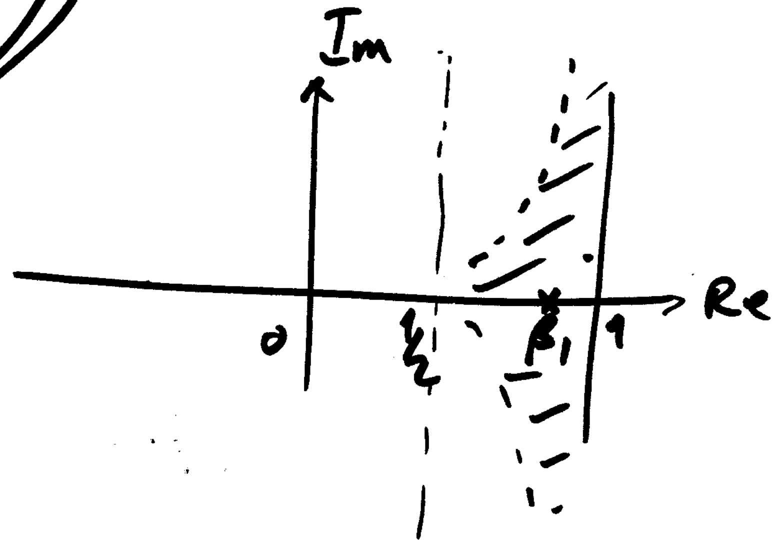
$$0 \leq \frac{1}{1-\beta_0}\left(\frac{1}{2} - \frac{2}{3}\right) + 2c_1 \log(4q)$$

$$\Downarrow$$

$$1-\beta_0 \geq \frac{1}{12c_1 \log(4q)}.$$

⑤ Then $L(s,\chi)$ has at most one zero β_1 in the region $\sigma > 1 - c/\log(4q)$ provided that $c < 1/12c_1$. □

This completes the proof! //



6

Theorem 14.5. Let χ be a non-principal character modulo q. Suppose that $c > 0$ is an absolute constant with the property that $L(s,\chi) \neq 0$ for $\sigma \geq 1 - c/\log(q\tau)$ where, in the case that χ is quadratic, we permit a possible exceptional zero $\beta \in \mathbb{R}$ with $\beta \leq 1$. Then whenever

$$\sigma \geq 1 - c/2\log(q\tau),$$

one has the following:

(i) when $L(s,\chi)$ has no exceptional ~~zero~~, ~~and~~ or ~~...~~ when β_1 is an exceptional zero of $L(s,\chi)$ ~~satisfying~~ and $|s-\beta_1| \geq 1/\log q$,

$$\frac{L'}{L}(s,\chi) \ll \log(q\tau), \quad \frac{1}{L(s,\chi)} \ll \log(q\tau) \text{ and } |\log L(s,\chi)| \leq \log\log(q\tau) + O(1);$$

(ii) when β_1 is an exceptional zero of $L(s,\chi)$ and $|s-\beta_1| \leq 1/\log q$, then

$$\frac{L'}{L}(s,\chi) = \frac{1}{s-\beta_1} + O(\log q) \quad (s \neq \beta_1), \quad |\arg L(s,\chi)| \leq \log\log q + O(1).$$

and

$$|s-\beta_1| \ll |L(s,\chi)| \ll |s-\beta_1|(\log q)^2.$$

(7) **Proof** (i) Easy when $\sigma \geq 1 + 1/\log(q\tau)$, since

$$\left| \frac{L'}{L}(s,\chi) \right| \leq \sum_{n=1}^{\infty} \Lambda(n) n^{-\sigma} = -\frac{\zeta'}{\zeta}(\sigma) \ll \frac{1}{\sigma - 1} \ll \log(q\tau).$$

In order to handle general $s = \sigma + it$ with

$$1 - c/2\log(q\tau) \leq \sigma \leq 1 + 1/\log(q\tau),$$

we seek to shift from $s_1 = 1 + 1/\log(q\tau) + it$ to s. We have

$$\frac{L'}{L}(s_1, \chi) \ll \log(q\tau),$$

so Lemma 14.2 yields

$$\sum_\rho \frac{1}{s_1 - \rho} + O(\log(q\tau)) = \frac{L'}{L}(s_1, \chi) \ll \log(q\tau),$$

where summation over ~~zeros~~ ρ of $L(s,\chi)$ with $\left|\rho - \left(\tfrac{3}{2} + it\right)\right| \leq 5/6$.

Hence $$\sum_\rho \frac{1}{s_1 - \rho} \ll \log(q\tau).$$

⑧ This implies

$$\sum_\rho \frac{1}{s-\rho} = \sum_\rho \left(\frac{1}{s-\rho} - \frac{1}{s_1-\rho} \right) + O(\log(q\tau)),$$

where for each ρ one has

$$\frac{1}{s-\rho} - \frac{1}{s_1-\rho} = \frac{1 + 1/\log(q\tau) - \sigma}{(s-\rho)(s_1-\rho)} \ll \frac{1}{\log(q\tau) \cdot |s_1-\rho|^2}$$

$$\ll \operatorname{Re}\left(\frac{1}{s_1-\rho} \right).$$

[Note: $\operatorname{Re}(s_1-\rho) = \operatorname{Re}\left((1 + 1/\log(q\tau) + it) - (1 - c/2\log(q\tau)) \right)$

$\gg 1/\log(q\tau)$ for $\rho \neq \beta_1$

and $\operatorname{Re}(s_1 - \beta_1) \geqslant 1/\log q$.]

Thus $$\sum_\rho \frac{1}{s-\rho} \ll \operatorname{Re}\left(\sum_\rho \frac{1}{s_1-\rho} \right) + O(\log(q\tau)).$$

Then by Lemma 14.2,

$$\frac{L'}{L}(s,\chi) = \sum_\rho \frac{1}{s-\rho} + O(\log(q\tau)) \ll \operatorname{Re}\left(\sum_\rho \frac{1}{s_1-\rho} \right) + O(\log(q\tau))$$

$$= \operatorname{Re}\left(\frac{L'}{L}(s_1,\chi) \right) + O(\log(q\tau)) \ll \log(q\tau). \square$$

16 Nov 2020 | **Recall:** **Theorem 14.5** Let $\chi \neq \chi_0$ be a character (mod q).

1 Let $c > 0$ be an absolute constant with $L(s, \chi) \neq 0$ for $\sigma \geq 1 - c/\log(q\tau)$, where, in case χ is quadratic, we permit a possible exceptional zero $\beta \in \mathbb{R}$ with $\beta < 1$. Then whenever $\sigma \geq 1 - c/2\log(q\tau)$, one has:

(i) When $L(s, \chi)$ has no exceptional zero, or when β_1 is an exceptional zero of $L(s, \chi)$ with $|s - \beta_1| \geq 1/\log q$,

$$\frac{L'}{L}(s, \chi) \ll \log(q\tau), \quad \frac{1}{L(s,\chi)} \ll \log(q\tau) \;\&\; |\log L(s,\chi)| \leq \log\log(q\tau) + O(1)$$

(ii) When β_1 is an exceptional zero of $L(s, \chi)$ and $|s - \beta_1| \leq \frac{1}{\log q}$, then

$$\frac{L'}{L}(s, \chi) = \frac{1}{s - \beta_1} + O(\log q) \qquad (s \neq \beta_1),$$

$$|\arg L(s, \chi)| \leq \log\log q + O(1) \qquad (s \neq \beta_1),$$

and

$$|s - \beta_1| \ll |L(s, \chi)| \ll (s - \beta_1)(\log q)^2.$$

② Proof:

(i) $$\frac{L'}{L}(s,\chi) \ll \log(q\tau) \quad \checkmark \quad \square$$

Next consider bound $|\log L(s,\chi)| \leq \log\log(q\tau) + O(1)$ of (i).

Again, when $\sigma \geq 1 + 1/\log(q\tau)$, we have

$$|\log L(s,\chi)| \leq \sum_{n=2}^{\infty} \frac{\Lambda(n)}{\log n} n^{-\sigma} = \log \zeta(\sigma) \leq \log\left(\frac{\sigma}{\sigma-1}\right)$$

$$\leq \log\log(q\tau).$$

We again put $s_1 = 1 + 1/\log(q\tau) + it$, and note ~~that~~ when $1 + 1/\log(q\tau) \geq \sigma \geq 1 - c/2\log(q\tau)$, we have

$$\log L(s,\chi) - \log L(s_1,\chi) = \int_{s_1}^{s} \frac{L'}{L}(z,\chi)\,dz$$

$$\ll |s - s_1| \log(q\tau) \ll 1$$

Hence $$|\log L(s,\chi)| \leq |\log L(s_1,\chi)| + O(1)$$

$$\leq \log\log(q\tau) + O(1). \quad \square$$

③ Then $\log\left(\frac{1}{|L(s,\chi)|}\right) = -\operatorname{Re}\left(\log L(s,\chi)\right)$

$$\leq \log\log(q\tau) + O(1)$$

$$\Rightarrow \frac{1}{|L(s,\chi)|} \ll \log(q\tau) \cancel{\ldots}.$$

(also $|L(s,\chi)| \ll \log(q\tau)$). □

(ii) Assume that $L(s,\chi)$ has an exceptional ~~zero~~ β_1 with $|s-\beta_1| \leq 1/\log q$. We may then suppose

$$1 - c/2\log(4q) \leq \sigma \leq 1 + 1/\log q,$$

so by Lemma 14·2,

$$\frac{L'}{L}(s,\chi) = \frac{1}{s-\beta_1} + \sum_{\rho}{}' \frac{1}{s-\rho} + O(\log q),$$

in which the summation over zeros ρ of $L(s,\chi)$ ~~satisfies~~ $|\rho - (\tfrac{3}{2}+it)| \leq \frac{5}{6}$. excluding β_1.

Thus, using the argument we applied in part (i),

④

$$\frac{L'}{L}(s,\chi) \ll \sum_{\rho}{}' \operatorname{Re}\left(\frac{1}{s_1-\rho}\right) + O(\log q).$$

$$+ \frac{1}{s-\beta_1} \qquad (s_1 = 1 + 1/\log q + it)$$

Hence

$$\frac{L'}{L}(s,\chi) = \frac{1}{s-\beta_1} + \operatorname{Re}\left(\frac{L'}{L}(s_1,\chi)\right) + O(\log q)$$

$$= \frac{1}{s-\beta_1} + O(\log q). \quad \square$$

For the final assertions of part (ii), we observe as in case (i) that

$$\log L(s,\chi) - \log L(s_1,\chi) = \int_{s_1}^{s} \frac{1}{z-\beta_1}\,dz + O(|s-s_1|(\log q))$$

$$= \log\left(\frac{s-\beta_1}{s_1-\beta_1}\right) + O(1),$$

Whence

⑤

$$\left| \log L(s,\chi) - \log\left(\frac{s-\beta_1}{s_1-\beta_1}\right)\right| \leq |\log L(s_1,\chi)| + O(1)$$

$$\leq \log\log q + O(1)$$

Note that $\arg(s-\beta_1) \ll 1$, $\arg(s_1-\beta_1) \ll 1$

and $\log|s_1-\beta_1| = -\log\log q + O(1)$. Thus

$$|\arg L(s,\chi)| \leq \log\log q + O(1)$$

and

$$\log\left|\frac{L(s,\chi)}{s-\beta_1}\right| \leq |\log|s_1-\beta_1|| + \log\log q + O(1)$$

$$= 2\log\log q + O(1)$$

&

$$\log\left|\frac{L(s,\chi)}{s-\beta_1}\right| \geq -\log|s_1-\beta_1| - \log\log q + O(1)$$

$$= O(1).$$

In conclusion

$$1 \ll \left|\frac{L(s,\chi)}{s-\beta_1}\right| \ll (\log q)^2. \quad \square$$

§ 15. The Prime Number Theorem in arithmetic progressions, I.

$$\pi(x;q,a) = \sum_{\substack{p\le x\\ p\equiv a \pmod q}} 1 , \quad \theta(x;q,a) = \sum_{\substack{p\le x\\ p\equiv a \pmod q}} \log p, \quad \psi(x;q,a) = \sum_{\substack{n\le x\\ n\equiv a \pmod q}} \Lambda(n)$$

$$\pi(x,\chi) = \sum_{p\le x} \chi(p) , \quad \theta(x,\chi) = \sum_{p\le x} \chi(p)\log p , \quad \psi(x,\chi) = \sum_{n\le x} \chi(n)\Lambda(n)$$

By orthog. of characters, we have, for example:

$$\psi(x;q,a) = \frac{1}{\phi(q)} \sum_{\chi\in X(q)} \bar{\chi}(a)\,\psi(x,\chi).$$

<u>Theorem 15·1.</u> There is a constant $c>0$ having the following property. Suppose ~~that~~ $q \le \exp(2c\sqrt{\log x})$. Then, when $L(s,\chi)$ has <u>no</u> exceptional zero, one has

$$\psi(x,\chi) = E_0(\chi)\, x + O(x\exp(-c\sqrt{\log x})).$$

⑦ Meanwhile, when $L(s,\chi)$ has an exceptional ~~character~~ zero β_1, one instead has

$$\psi(x,\chi) = -\frac{x^{\beta_1}}{\beta_1} + O\left(x \exp\left(-c\sqrt{\log x}\right)\right)$$

<u>Proof:</u> By our explicit version of Perron's formula,

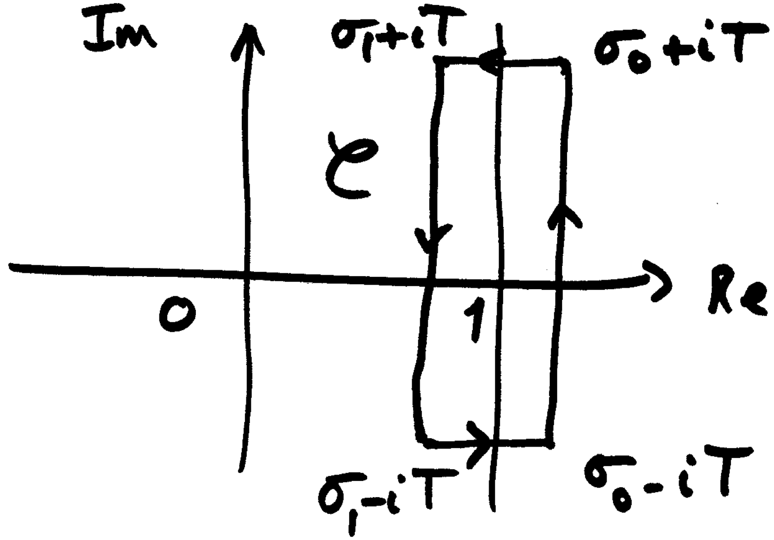
$$\psi(x,\chi) = -\frac{1}{2\pi i}\int_{\sigma_0 - iT}^{\sigma_0 + iT} \frac{L'}{L}(s,\chi)\frac{x^s}{s}\,ds + R(T),$$

where $\sigma_0 > 1$ and

$$R(T) \ll \sum_{\frac{x}{2} < n < 2x} \Lambda(n) \min\left\{1, \frac{x}{T|x-n|}\right\} + \frac{(4x)^{\sigma_0}}{T}\sum_{n=1}^{\infty}\frac{\Lambda(n)}{n^{\sigma_0}}.$$

As in proof of Theorem 11.1, we can suppose $2 \le T \le x$ and put $\sigma_0 = 1 + 1/\log x$, giving $R(T) \ll \frac{x}{T}(\log x)^2$.

(8) Path $\mathcal{C}$ of contour as in Theorem 11·1, but be careful to account for poss. exceptional zero β_1.



(i) Suppose that there is no exceptional zero.

We take $\sigma_1 = 1 - \dfrac{c_1}{5\log(qT)}$, where $c_1>0$ is any constant s.t. $L(s,\chi)$ has no zeros in $\sigma > 1 - \dfrac{c_1}{\log qT}$ (permitted by Thm 14·4).

When $\chi \neq \chi_0$, follows that

$$-\frac{L'}{L}(s,\chi)\cdot\frac{x^s}{s}$$

is analytic inside and on $\mathcal{C}$, and same when $\chi = \chi_0$ except for simple pole at $s=1$ with residue x.

18 Nov 2020 ①

Recall: **Theorem 15.1** There is a constant $c>0$ having the following property. Suppose that $q \leq \exp(2c\sqrt{\log x})$. Then

$$\psi(x,\chi) = E_0(\chi)\, x + O(x \exp(-c\sqrt{\log x}))$$

($L(s,\chi)$ has no exceptional zero)

and

$$\underbrace{\psi(x,\chi)}_{=\sum_{n\leq x} \Lambda(n)\chi(n)} = -\frac{x^{\beta_1}}{\beta_1} + O(x\exp(-c\sqrt{\log x}))$$

($L(\beta_1,\chi)=0$ with β_1 exceptional).

Proof. Apply Perron's formula:

$$\psi(x,\chi) = -\frac{1}{2\pi i}\int_{\sigma_0-iT}^{\sigma_0+iT} \frac{L'}{L}(s,\chi)\frac{x^s}{s}\,ds + R(T),$$

where $\sigma_0>1$ and

$$R(T) \ll \sum_{\frac{x}{2}<n<2x} \Lambda(n)\min\left\{1, \frac{x}{T|x-n|}\right\} + \frac{(4x)^{\sigma_0}}{T}\sum_{n=1}^{\infty}\frac{\Lambda(n)}{n^{\sigma_0}}.$$

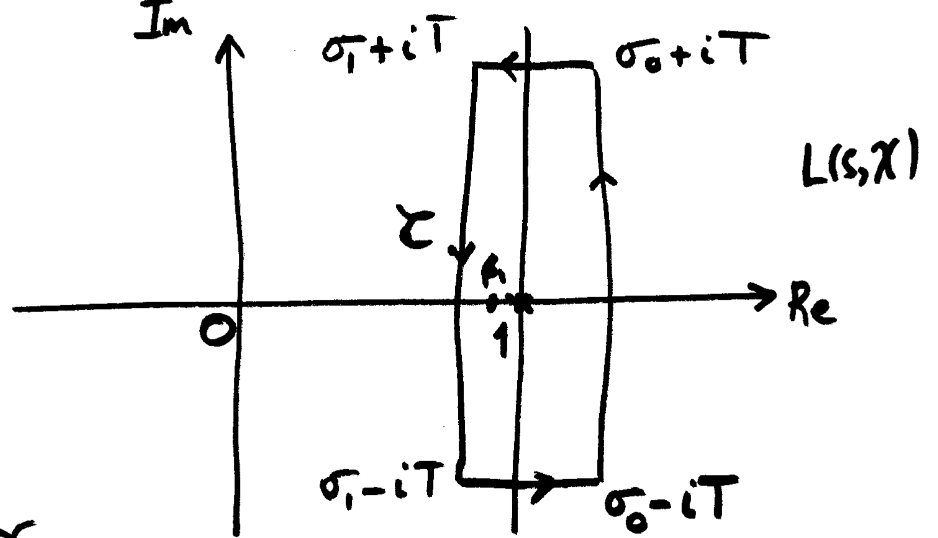
Suppose $2 \leq T \leq x$ & put $\sigma_0 = 1 + 1/\log x \longrightarrow R(T) \ll \frac{x}{T}(\log x)^2$

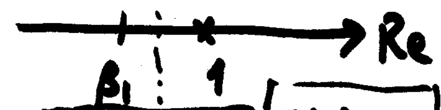
②

Suppose (as we may) that $L(s,\chi)$ has no zeros in the region

$$\sigma > 1 - \frac{c_1}{\log(qT)} \quad \text{(Thm 14.4)}$$

(suitable $c_1 > 0$) except possibly for the exceptional zero β_1 when χ is quadratic.





$$\boxed{\beta_1 < 1 - c(\varepsilon)/q^{\varepsilon}}$$

(i) Suppose that there is no exceptional zero.

We take $\sigma_1 = 1 - \frac{c_1}{5\log(qT)}$ ($\mathcal{C}$ lies inside zero-free region).

When $\chi \neq \chi_0$, then $-\frac{L'}{L}(s,\chi)\frac{x^s}{s}$ is analytic inside/on $\mathcal{C}$, and same when $\chi = \chi_0$ except for a simple pole at $s=1$ with residue x.

Note: $-\frac{L'}{L}(s,\chi_0) = -\frac{\zeta'}{\zeta}(s) - \sum_{p \mid q} \frac{\log p}{p^s - 1} = \frac{1}{s-1} + O(1)$ as $s \to 1$

③ Follow the arguments of proof of Theorem 11·1 to handle ~~the~~ left vertical and horizontal segments of $\mathcal{C}$ using Theorem 14·5 (i). Thus

$$\psi(x,\chi) - E_0(\chi)x \ll x(\log x)^2\left(\frac{1}{T} + \exp\left(-\frac{c_1 \log x}{5\log(qT)}\right)\right)$$

Taking $T = \exp(2c\sqrt{\log x})$ and assuming ~~that~~ $q \le \exp(2c\sqrt{\log x})$ with $c = \sqrt{c_1/40}$, we find that

$$\psi(x,\chi) - E_0(\chi)x \ll x(\log x)^2\left(\exp(-2c\sqrt{\log x}) + \exp\left(-c_1\frac{\log x}{5\cdot 4c\sqrt{\log x}}\right)\right)$$

$$\ll x\exp(-c\sqrt{\log x}). \quad \square$$

$$\left[\left((\log x)^2 \exp(-2c\sqrt{\log x}) = \exp(2\log\log x - 2c\sqrt{\log x})\right)\right]$$

<u>(ii) Suppose ~~that~~ there is an exceptional ~~zero~~ β_1 and $\beta_1 \ge 1 - c_1/4\log(qT)$.</u>

④

In this case, take $\sigma_1 = 1 - c_1 / 3\log(qT)$.

In this case there is a pole of

$$-\frac{L'}{L}(s, \chi_1)\frac{x^s}{s} \qquad (\text{NB: } \chi \neq \chi_0)$$

at $s = \beta_1$ inside $\mathcal{C}$ having residue $-\frac{x^{\beta_1}}{\beta_1}$.

$$\left[-\frac{L'}{L}(s, \chi) = -\frac{1}{s-\beta_1} + O(1) \quad \text{as } s \to \beta_1 \right].$$

The remaining analysis of (i) applies to give:

$$\psi(x, \chi) = -\frac{x^{\beta_1}}{\beta_1} + O\left(x(\log x)^2 \left(\frac{1}{T} + \exp\left(-c_1 \frac{\log x}{3\log(qT)}\right)\right)\right)$$

$$= -\frac{x^{\beta_1}}{\beta_1} + O(x \exp(-c\sqrt{\log x})). \quad \square$$

(iii) <u>Suppose that there is an exceptional zero β_1 and</u>

$$\underline{\beta_1 < 1 - \frac{c_1}{4\log(qT)}}.$$

In this case again take $\sigma_1 = 1 - c_1/5\log(qT)$, and

(5) follow argument of (i). In this case β_1 does <u>not</u> lie in or on $\mathcal{C}$, and we again obtain

$$\psi(x,\chi) - \underbrace{E_0(\chi)}_{\shortparallel \atop 0} x \ll x(\log x)^2\left(\frac{1}{T} + \exp\left(-c_1 \frac{\log x}{5\log(qT)}\right)\right)$$

Note that: $$\frac{-x^{\beta_1}}{\beta_1} \ll \exp(\beta_1 \log x) < x\exp\left(-\frac{c_1 \log x}{4\log(qT)}\right)$$

Thus $$\psi(x,\chi) = \frac{-x^{\beta_1}}{\beta_1} + O\left(x\exp\left(-c\sqrt{\log x}\right)\right)$$

assuming $q \le \exp(2c\sqrt{\log x})$, $T = \exp(2c\sqrt{\log x})$. $\square$ //

$$\psi(x;q,a) = \frac{1}{\phi(q)} \sum_{\chi \in X(q)} \bar{\chi}(a)\,\psi(x,\chi)$$

$$= \frac{x}{\phi(q)} - \sum_{\beta \text{ exc.}} \frac{\chi_\beta(a)\, x^\beta}{\phi(q)\,\beta} + O(x\exp(-c\sqrt{\log x}))$$

⑥

<u>Theorem 15.3</u> (Landau) There is a constant $c > 0$ with the following property. Suppose ~~that~~ χ_i is a quadratic character modulo q_i $(i = 1, 2)$, and further suppose $\chi_1 \chi_2$ is non-principal. Then $L(s, \chi_1) L(s, \chi_2)$ has at most one real zero β with

$$1 - \frac{c}{\log(q_1 q_2)} < \beta \leq 1.$$

20 Nov 2020

①

Recall : Primes in a.p.

<u>Theorem 15.1.</u> There exists $c>0$ s.t. when $q \le \exp(2c\sqrt{\log x})$, one has

- $\psi(x,\chi) = E_0(\chi)\, x + O(x \exp(-c\sqrt{\log x}))$

(no exceptional zero),

$\psi(x,\chi) = \sum_{n \le x} \Lambda(n)\chi(n)$

- $\psi(x,\chi) = -\dfrac{x^{\beta_1}}{\beta_1} + O(x \exp(-c\sqrt{\log x}))$,

($L(s,\chi)$ has exceptional zero β_1)

↑ quadratic.

$$\psi(x;q,a) = \frac{1}{\varphi(q)} \sum_{\chi \in X(q)} \bar{\chi}(a)\, \psi(x,\chi)$$

$$\Downarrow$$

$$\psi(x,\chi) = \frac{x}{\varphi(q)} + O(x\exp(-c\sqrt{\log x}))$$

unless there are exceptional zeros — but how many?

②

Lemma 15.2 (Landau) Suppose that χ_1 and χ_2 are quadratic characters. Then whenever $\sigma > 1$, one has

$$-\frac{\zeta'}{\zeta}(\sigma) - \frac{L'}{L}(\sigma, \chi_1) - \frac{L'}{L}(\sigma, \chi_2) - \frac{L'}{L}(\sigma, \chi_1\chi_2) \geq 0.$$

Proof. The lhs can be written $\sum_{n=1}^{\infty} a_n \Lambda(n) n^{-\sigma}$, where

$$\begin{aligned} a_n &= 1 + \chi_1(n) + \chi_2(n) + \chi_1\chi_2(n) \\ &= (1 + \chi_1(n))(1 + \chi_2(n)) \geq 0. \end{aligned}$$

The desired conclusion follows from abs. conv. //

③

<u>Theorem 15.3</u> (Landau) There is a constant $c > 0$ with the following property. Suppose that χ_i is a quadratic character modulo q_i for $i = 1, 2$, and further $\chi_1\chi_2$ is non-principal. Then $L(s, \chi_1)L(s, \chi_2)$ has at most one real zero β with

$$1 - \frac{c}{\log(q_1q_2)} < \beta < 1.$$

χ_0 modulo $10q_1$

$\chi_0\chi_1(n)$

$\chi_0\chi_1^2(n)$

<u>Proof.</u> Thm 14.4 shows that, should any zero β of the type exist, then ~~it is associated~~ there is at most one associated with $L(s, \chi_1)$, and at most one associated with $L(s, \chi_2)$.

Suppose that <u>both</u> $L(s, \chi_1)$ & $L(s, \chi_2)$ have such zeros β_1 and β_2 resp. There is no loss of generality that we have $5/6 \leq \beta_i \leq 1$ $(i = 1, 2)$. Also, can suppose that $\chi_1\chi_2$ is non-principal. Then (Lemma 14.2),

(4) When $0 < \delta \le 1$, and for a suitable constant $c_1 > 0$,

$$-\frac{\zeta'}{\zeta}(1+\delta) \le \frac{1}{\delta} + c_1 \log q_1 q_2 ,$$

$$-\frac{L'}{L}(1+\delta, \chi_i) \le -\frac{1}{1+\delta-\beta_i} + c_1 \log q_i \quad (i=1,2)$$

$$-\frac{L'}{L}(1+\delta, \underset{\substack{\uparrow \\ \chi_1\chi_2 \ne \chi_0}}{\chi_1\chi_2}) \le \underset{\uparrow}{} c_1 \log(q_1 q_2).$$

$$-\sum_{\rho} \frac{1}{1+\delta-\rho} \qquad L(\rho, \chi_1\chi_2) = 0$$

Subst. into Lemma 15.2, we get

$$0 \le \frac{1}{\delta} - \frac{1}{1+\delta-\beta_1} - \frac{1}{1+\delta-\beta_2} + 3c_1 \log(q_1 q_2)$$

If we suppose $\beta_1 \le \beta_2$ and put $\delta = 2(1-\beta_1)$, it follows that

$$0 \le \frac{1}{\delta} - \frac{2}{1+\delta-\beta_1} + 3c_1 \log(q_1 q_2)$$

⑤

whence

$$0 \leq \frac{1}{1-\beta_1}\left(\frac{1}{2} - \frac{2}{3}\right) + 3c_1 \log(q_1 q_2)$$

$$\Rightarrow \quad 1 - \beta_1 \geq \frac{1}{18 c_1 \log(q_1 q_2)}.$$

We therefore conclude that whenever $c < 1/18c_1$, then $L(s,\chi_1)L(s,\chi_2)$ has at most one real zero β with $1-\beta > c/\log(q_1 q_2)$. //

⑥ <u>Corollary 15.4</u> (Landau) There is a constant $c>0$ with the property ~~that~~ $\prod_{\chi \in X(q)} L(s,\chi)$ has at most one zero in the region $\sigma > 1 - c/\log(q\tau)$. If such a zero exists, then it is necessarily real, and ~~the~~ associated character is quadratic.

<u>Proof.</u> Suppose, by way of contradiction, ~~that~~ there are two zeros (or more) in $\sigma > 1 - c/\log(q\tau)$. As a consequence of Theorem 14.4, when c is suff. small, this is impossible unless both zeros are real and associated characters χ_1 and χ_2 quadratic. But ~~then~~ Theorem 15.3 applies to give a contradiction.

Application to primes in a.p.

(7) <u>Theorem 15.5.</u> (Page) There is a constant $c>0$ having the ~~following~~ property. Suppose ~~that~~ $q \leq \exp(2c\sqrt{\log x})$ and $a \in \mathbb{Z}$ ~~and a~~ satisfy $(a,q)=1$. Then:

(i) When there is no exceptional character modulo q, one has

$$\psi(x;q,a) = \frac{x}{\phi(q)} + O(x\exp(-c\sqrt{\log x}));$$

(ii) When there is an exceptional χ_1 modulo q, and β_1 is the associated exceptional zero of $L(s,\chi_1)$, one has instead

$$\psi(x;q,a) = \frac{x}{\phi(q)} - \frac{\chi_1(a)x^{\beta_1}}{\phi(q)\beta_1} + O(x\exp(-c\sqrt{\log x})).$$

<u>Proof.</u> Have $\psi(x;q,a) = \frac{1}{\phi(q)} \sum_{\chi \in X(q)} \bar{\chi}(a)\psi(x,\chi)$,

so desired formulae follow from Theorem 15.1.

⑧ Two notes: (i) There is at most one exceptional character / zero (from Cor. 15·4).

(ii) When $q > \exp(2c\sqrt{\log x})$, note that claimed asymptotic formulae are worse than "trivial":

$$\psi(x;q,a) \le \underbrace{\left(\frac{x}{q}+1\right)}_{\#\, n \equiv a \pmod q} \log x \ll x \exp(-2c\sqrt{\log x}) \cdot \log x$$

no larger than error terms in our formulae. //

<u>To do:</u> Siegel: $L(1,\chi) > c_1(\varepsilon)\, q^{-\varepsilon}$

$$\Downarrow$$

$$\beta_1 < 1 - c_2(\varepsilon)\, q^{-\varepsilon}.$$

23 Nov 2020

①

Recall: Theorem 15.5 (Page) There is a constant $c > 0$ having the following property. Suppose that $q \leq \exp(2c\sqrt{\log x})$ and $a \in \mathbb{Z}$ satisfy $(a,q) = 1$.

(i) When there is no exceptional character modulo q, one has

$$\psi(x;q,a) = \frac{x}{\phi(q)} + O\left(x \exp(-c\sqrt{\log x})\right);$$

(ii) When there is an exceptional character χ_1 modulo q, and β_1 is the associated exceptional zero of $L(s,\chi_1)$, then

$$\psi(x;q,a) = \frac{x}{\phi(q)} - \frac{\chi_1(a)x^{\beta_1}}{\phi(q)\beta_1} + O\left(x \exp(-c\sqrt{\log x})\right).$$

Aim of final part of course:

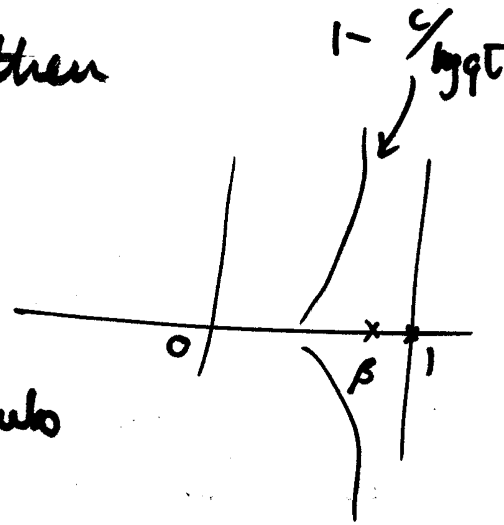
Theorem 15.6 (Siegel) For each $\varepsilon > 0$, there is a positive constant $C_1(\varepsilon)$ such that, when χ is a quadratic character modulo q, then

$$L(1,\chi) > C_1(\varepsilon) q^{-\varepsilon}.$$

Proof: Later. //

(2)

Corollary 15.7. For any $\varepsilon > 0$, there is a positive constant $C_2(\varepsilon)$ such that, when χ is a quadratic character modulo q and β is a real zero of $L(s,\chi)$, then



$$\beta < 1 - C_2(\varepsilon)\, q^{-\varepsilon}.$$

Proof: We may suppose that there is a const. $c > 0$ s.t. there is at most one character χ modulo q (quadratic) having a real ~~zero~~ β ~~satisfying~~

$1 - c/\log q < \beta < 1$. If such a zero exists, then from Theorem 14.5 (ii) that

$$L(1,\chi) \ll (1-\beta)(\log q)^2.$$

Then since $L(1,\chi) > C_1(\varepsilon/2)\, q^{-\varepsilon/2}$, it follows that there is a constant $c_3 > 0$ with property that

$$C_3(1-\beta)(\log q)^2 > L(1,\chi) > C_1(\varepsilon/2)\, q^{-\varepsilon/2}$$

$$\wedge$$

$$C_3^2(1-\beta)\, q^{\varepsilon/2} \quad \Rightarrow \quad 1-\beta > C_2(\varepsilon)\, q^{-\varepsilon}, \text{ where } C_2(\varepsilon) = C_1(\varepsilon/2)/C_3^2.$$

③

<u>Theorem 15.8</u> There is a constant $c>0$ having the following property. Let $A>0$ be fixed. Then whenever $x \geq x_0(A)$ (i.e. when x is sufficiently large in terms of A). and $q \leq (\log x)^A$, one has

$$\psi(x,\chi) = E_0(\chi)\, x + O\left(x \exp\left(-c\sqrt{\log x}\right)\right).$$

<u>Proof.</u> This is a consequence of Theorem 15.1. All that is required is to handle the possible exc. zero. β_1 of $L(s,\chi)$ for a quadratic character χ. In such a situation, we have

$$\psi(x,\chi) = -\frac{x^{\beta_1}}{\beta_1} + O\left(x \exp\left(-c_1\sqrt{\log x}\right)\right)$$

$(\text{some } c_1 > 0)$,

where (from Corollary 15.7), we may suppose ~~that~~ for some $c_2(\varepsilon)>0$, we have $\beta_1 < 1 - c_2(\varepsilon)\, q^{-\varepsilon}$.

But

$$x^{\beta_1} = x \exp\left(-(1-\beta_1)\log x\right) \leq x \exp\left(-c_2(\varepsilon)\, q^{-\varepsilon}\log x\right)$$

④

$$\leq x \exp\left(- c_2(\varepsilon) (\log x)^{1 - A\varepsilon} \right).$$

If we take $\varepsilon = 1/(3A)$, then we deduce

$$\frac{x^{\beta_1}}{\beta_1} \leq 2x \exp\left(- c_2(\varepsilon) (\log x)^{2/3} \right)$$

$$< x \exp\left(- c_1 \sqrt{\log x} \right)$$ when $x \geq x_0(A)$, suitable $x_0(A)$.

Thus

$$\psi(x, \chi) = O\left(x \exp\left(- c_1 \sqrt{\log x} \right) \right).$$ //

<u>Corollary 15.9</u> (Siegel - Walfisz theorem) There is a constant $c > 0$ with the following property. Let $A > 0$ be fixed. Then whenever $x \geq x_0(A)$ and $q \leq (\log x)^A$, and $(a, q) = 1$, then

$$\psi(x; q, a) = \frac{x}{\varphi(q)} + O_A\left(x \exp\left(- c\sqrt{\log x} \right) \right).$$

<u>Proof:</u> Use orthogonality of characters. //

§ 16. Primitive Dirichlet characters and Gauss sums.

$$L(s,\chi_0) = \zeta(s) \prod_{p|q} (1-p^{-s}).$$

<u>Definition 16·1.</u> Suppose ~~that~~ $d|q$, ~~that~~ χ^* is a Dirichlet character modulo d, and χ is a Dirichlet character modulo q. Then we say ~~that~~ χ^* <u>induces</u> χ when

$$\chi(n) = \begin{cases} \chi^*(n), & \text{when } (n,q)=1, \\ 0, & \text{when } (n,q)>1. \end{cases}$$

Thus, if χ_0 is ~~the~~ principal character modulo q, we have $\chi = \chi^* \chi_0$. Notice ~~that~~ $\chi(n) \neq \chi^*(n)$ whenever $(n,q)>1$ and $(n,d)=1$. Moreover,

$$L(s,\chi) = L(s,\chi^*) \prod_{p|q} \left(1 - \frac{\chi^*(p)}{p^s}\right)$$

6

Definition 16.2. Let χ be a Dirichlet character modulo q. We say that d is a quasi period of χ if $\chi(m) = \chi(n)$ whenever $m \equiv n \pmod d$ and $(mn, q) = 1$.

Note that the least quasiperiod of χ is a divisor of q. For if d is a quasiperiod of χ, one can show that $e = (d, q) \mid q$ is also a quasiperiod of χ. To see this, whenever $m \equiv n \pmod e$ and $(mn, q) = 1$, then for some $l \in \mathbb{Z}$ one has

$$m - n = le = l(d, q) = l(dx + qy), \text{ some } x, y \in \mathbb{Z}.$$

Thus $\chi(m) = \chi(m - l \cdot y \cdot q) = \chi(n + lx \cdot d) = \chi(n)$, so that e is indeed a quasi-period of χ.

Similar argument shows that when d_1 and d_2 are both quasi-periods of χ, then so too is (d_1, d_2), so the least quasi-period of χ divides all other quasi-periods (and q).

<u>Definition 16.3.</u> (i) The least quasi-period d of a Dirichlet character is called the <u>conductor</u> of χ.

(ii) A character χ modulo q is called <u>primitive</u> when q is the <u>least quasi-period</u> of χ.

conductor

30 NOV 2020

1

Recall: We aim to prove:

Theorem 15.6 (Siegel) For each $\varepsilon > 0$, there is a positive constant $c_1(\varepsilon)$ such that, when χ is a quadratic character modulo q, then $L(1,\chi) > c_1(\varepsilon)\, q^{-\varepsilon}$.

$$\Downarrow$$

Corollary 15.7. There exists $c_2(\varepsilon) > 0$ such that, when χ is a quadratic character modulo q, and β is a real zero of $L(s,\chi)$, then $\beta < 1 - c_2(\varepsilon) q^{-\varepsilon}$.

$$\Downarrow$$

Corollary 15.9 (Siegel - Walfisz theorem)
There exists $c > 0$ s.t., when $A > 0$ fixed, $x \geq x_0(A)$ and $q \leq (\log x)^A$, and $(a,q) = 1$, then

$$\psi(x;q,a) = \frac{x}{\varphi(q)} + O_A\left(x \exp(-c\sqrt{\log x})\right).$$

§ 16. Primitive Dirichlet characters.

χ^* induces χ

mod d mod q

$d | q$

$$\chi(n) = \begin{cases} \chi^*(n) & (n,q)=1 \\ 0 & (n,q)>1. \end{cases}$$

$\chi = \chi_0 \chi^*$ $L(s,\chi) \longleftrightarrow L(s,\chi^*)$

χ mod q Say d is a quasi-period of χ if $\chi(m) = \chi(n)$ whenever $m \equiv n \pmod d$ and $(mn, q) = 1$.

- quasiperiod of χ mod q is a divisor of q
- least quasiperiod of χ divides all other quasiperiods, and q.

Definition 16·3. (i) The least quasi-period d of a Dirichlet character χ is called the conductor of χ.

(ii) A character χ modulo q is called primitive when q is the least quasi-period of χ.

3 **Theorem 16.4** Let χ be a Dirichlet character modulo q, and let d be the conductor of χ. Then $d \mid q$, and there is a unique primitive character χ^* modulo d that induces χ.

[If χ_0 is principal char. mod q, then $\chi = \chi_0 \chi^*$].

Proof: We have seen already that the least quasiperiod of χ, namely d, is a divisor of q. We begin by showing that χ is induced by some character χ^* modulo d.

Suppose that $n \in \mathbb{Z}$ satisfies $(n, d) = 1$. Let

$$d_0 = (d^\infty, q) := \lim_{m \to \infty} (d^m, q).$$

and put $r = q/d_0$, so that $(r, d) = 1$. Then there exists $k \in \mathbb{Z}$ with $(n + kd, r) = 1$, whence

$$1 = (n + kd, r d_0) = (n + kd, q).$$

4 For any such integer k, we define

$$\chi^*(n) := \chi(n + kd).$$

Note: since d is a quasi-period of χ, then the choice of k does not impact the definition of $\chi^*(n)$.

This defines $\chi^*(n)$ when $(n,d)=1$, and we put $\chi^*(n) = 0$ when $(n,d) > 1$.

Plainly, χ^* has period d. Also, when $(n,m)=1$, then for suitable integers u and v ~~that~~ we have ~~that~~

$$\begin{aligned} \chi^*(n)\chi^*(m) &= \chi(n+ud)\chi(m+vd) \\ &= \chi(nm + (um + vn + uvd)d) \\ &= \chi^*(nm) \quad (\text{NB coprimality used here) } (nm,d)=1. \end{aligned}$$

Then χ^* is multiplicative with period d and satisfies $\chi^*(n) = 0$ for $(n,d)>1$, and hence χ^* is a Dirichlet char. modulo d.

⑤ Indeed, if χ_0 is the principal character modulo q, one has

$$\chi(n) = \chi_0(n)\,\chi^*(n),$$

so χ^* induces χ.

Observe that <u>χ^* is primitive</u>. If χ^* were to have a quasi-period smaller than d, then so too would χ, contradicting the minimality of d. To see that χ^* is unique, observe that if some other character χ^+ modulo d also induces χ, then for any $n \in \mathbb{Z}$ with $(n, d) = 1$, we could find $k \in \mathbb{Z}$ with

$$\chi^*(n) = \chi^*(n+kd) \underset{(n+kd,q)=1}{=} \chi(n+kd) = \chi^+(n+kd) = \chi^+(n),$$

so $\chi^* = \chi^+$. //

<u>Lemma 16.5.</u> Suppose that $q_1, q_2 \in \mathbb{N}$ and $(q_1, q_2) = 1$. Suppose that χ_1 and χ_2 are characters modulo q_1 and q_2 respectively, and put

$$\chi(n) = \chi_1(n)\chi_2(n).$$

Then the character χ is primitive modulo q_1q_2 if and only if both χ_1 and χ_2 are primitive.

<u>Proof.</u> ($\Rightarrow$) Put $q = q_1q_2$, and suppose χ primitive modulo q. Also, write $d_i := \mathrm{cond}(\chi_i)$ $(i=1,2)$. Seek to show that $q = d_1d_2$, whence $d_1 = q_1$ and $d_2 = q_2$, so χ_1 & χ_2 are both primitive.

To see this, observe that whenever $m \equiv n \pmod{d_1d_2}$ and $(mn, q) = 1$, one has $\chi_i(m) = \chi_i(n)$ $(i = 1,2)$, whence $\chi(m) = \chi(n)$. Thus d_1d_2 is a quasi-period

of χ, so by primitivity of χ, hence $q = d_1 d_2$. $\square$
(So χ_1 and χ_2 are both primitive).

($\Leftarrow$) Suppose χ_i is primitive modulo q_i $(i=1,2)$, and let $d = \text{cond}(\chi)$. Aim to show that $d = q$, whence χ is primitive. Put $d_i = (d, q_i)$ $(i=1,2)$. Then whenever $m \equiv n \pmod{d_1}$ and $(mn, q_1) = 1$, we may choose integers $m' \equiv m \pmod{q_1}$ and $n' \equiv n \pmod{q_1}$ s.t. $m' \equiv n' \equiv 1 \pmod{q_2}$. We have $m' \equiv n' \pmod{d_1 d_2}$, that is $m \equiv n \pmod{d}$ and also $(m'n', q) = 1$.

Hence

$$\begin{array}{ccc} \chi(m') & = & \chi(n') \\ \| & & \| \\ \chi_1(m')\chi_2(m') & & \chi_1(n')\chi_2(n') \\ \| & & \| \\ \chi_1(m)\chi_2(1) & & \chi_1(n)\chi_2(1) \end{array}$$

8

Thus $x_1(m) = x_1(n)$, and so d_1 is a quasiperiod of x_1. Similarly, d_2 is a quasi-period of x_2. Then by primitivity, have $d_1 = q_1$ and $d_2 = q_1$, so $q_1 q_2 = d_1 d_2 = d$, showing ~~that~~ x is primitive. $\square$ //

02 Dec 2020

① **Recall:** We were discussing primitive characters.

Lemma 16.5 Suppose $q_1, q_2 \in \mathbb{N}$ and $(q_1, q_2) = 1$. Suppose also χ_1, χ_2 characters modulo q_1, q_2, resp., and put $\chi(n) = \chi_1(n)\chi_2(n)$. Then

χ primitive modulo $q_1 q_2$ $\Leftrightarrow$ χ_1 & χ_2 primitive $\begin{pmatrix}\text{mod } q_1 \\ \phantom{\text{mod }} q_2\end{pmatrix}$.

Suffices to understand primitive characters modulo p^h for $p^h \| q$.

②

---

p odd: $q = p^h$

$$\chi(n) = e\left(\frac{k \operatorname{ind}_g n}{\varphi(p^h)}\right) \quad \boxed{(n,p)=1}$$ for g primitive root modulo p^h

h = 1 χ primitive if and only if $\chi \neq \chi_0$, so $(p-1) \nmid k$.

h > 1 χ primitive if and only if $p \nmid k$.

② p even $(p = 2)$

$(q = 2^h)$ $h = 1$: $\chi = \chi_0$ not primitive

$h = 2$: χ either principal (not primitive) or not, in which case $\chi(4k+1) = 1$ and $\chi(4k-1) = -1$, and ~~this~~ is primitive.

$h \geq 3$: $\chi(n) = e\left(\frac{ju}{2} + \frac{kv}{2^{h-2}}\right)$

with $n \equiv (-1)^u 5^v \pmod{2^h}$

χ primitive if and only if k odd.

③

§17. Proof of Siegel's Lemma.

Goal : χ quadratic (mod q.) $\Rightarrow$ $L(1,\chi) \gg_{\varepsilon} q^{-\varepsilon}$.

Recall : (Dirichlet's theorem) $L(1,\chi) \neq 0$.

④

Lemma 4.1 (Estermann) Suppose that $f(s)$ is analytic for $|s-2| \leq \frac{3}{2}$, and that $|f(s)| \leq M$ for s lying in this disc. Put $F(s) = \zeta(s) f(s)$, [Suppose $F(s)$ can be written as a Dirichlet series], and define the coefficients $r(n)$ $(n \in \mathbb{N})$ by means of the relation

$$F(s) = \sum_{n=1}^{\infty} r(n) n^{-s} \qquad (\sigma > 1).$$

Suppose in addition that $r(1) = 1$ and $r(n) \geq 0$ $(n \in \mathbb{N})$. Then, whenever there exists $\sigma \in [19/20, 1)$ such that $f(\sigma) \geq 0$, one has

$$f(1) \geq \frac{1}{4}(1-\sigma) M^{-3(1-\sigma)}.$$

Proof: We expand $F(s)$ as a power series in $(2-s)$. Thus we have

$$F(s) = \sum_{k=0}^{\infty} b_k (2-s)^k,$$

⑤ for $|s-2|<1$, say. By Cauchy's formula, we see that

$$b_k = \frac{(-1)^k}{k!} F^{(k)}(2) = \frac{1}{k!} \sum_{n=1}^{\infty} r(n) n^{-2} (\log n)^k .$$

Then our hypotheses on $r(n)$ ensure that $b_k \geq 0$ for all k, and moreover

$$b_0 = \sum_{n=1}^{\infty} r(n) n^{-2} \geq 1 .$$

On noting that

$$\frac{1}{s-1} = \frac{1}{1-(2-s)} = \sum_{k=0}^{\infty} (2-s)^k$$

$$(|s-2|<1),$$

we obtain

$$F(s) - \frac{f(1)}{s-1} = \sum_{k=0}^{\infty} (b_k - f(1))(2-s)^k .$$

$$(|s-2|<1)$$

$$\left(f(s)\zeta(s) - \frac{f(1)}{s-1} \right)$$

(17.1).

⑥ The lhs of this relation is analytic for $|s-2| \le 3/2$, so the series on rhs converges in this disc (goes beyond $|s-2| < 1$).

Next, bound the coefficients on rhs, in particular $b_k - f(1)$ — by bounding lhs of (17.1).

Observe ~~that~~ when $|s-2| = 3/2$, we have $|s-1| \ge 1/2$, $|s| \le 7/2$ and $\sigma \ge \frac{1}{2}$. Also,

$$|\zeta(s)| \le \left| 1 + \frac{1}{s-1} + s\int_1^\infty \frac{[u]-u}{u^{s+1}}\, du \right|$$

$$\le 1 + \frac{1}{|s-1|} + \frac{|s|}{\sigma} \le 1 + 2 + \frac{7/2}{1/2}$$

$$\le 10.$$

Also $|f(s)| \le M$, so

⑦

$$\left| F(s) - \frac{f(1)}{s-1} \right| \leq |\zeta(s) f(s)| + \left| \frac{f(1)}{s-1} \right|$$

$$\leq 10M + \frac{M}{1/2} = 12M.$$

Then by the Cauchy coefficient inequalities,

$$|b_k - f(1)| \leq \left(\sup_{|s-2| = 3/2} \left| F(s) - \frac{f(1)}{s-1} \right| \right) \Big/ \left(\tfrac{3}{2}\right)^k$$

$$\leq 12M \left(\tfrac{2}{3}\right)^k.$$

We may now bound rhs of (17.1). Truncate infinite series at K. Thus, whenever $\frac{1}{2} < \sigma \leq 2$, we obtain

$$\underset{\substack{\shortparallel \\ F(\sigma)}}{\zeta(\sigma) f(\sigma)} - \frac{f(1)}{\sigma - 1} \geq \sum_{k=0}^{K} (b_k - f(1))(2-\sigma)^k - 12M \sum_{k>K} \left(\tfrac{2}{3}(2-\sigma) \right)^k. \quad \text{— (17.2)}$$

Dec 2020 Recall: Aim to prove Siegel's lemma.

Theorem 15.6 For each $\varepsilon > 0$, there is a positive constant $C_1(\varepsilon)$ such that, when χ is a quadratic character modulo q, then

$$L(1,\chi) > C_1(\varepsilon) q^{-\varepsilon}.$$

$$\Downarrow$$

Siegel-Walfisz theorem: When $(a,q)=1$, one has

$$\psi(x;q,a) = \frac{x}{\varphi(q)} + O_A\left(x \exp(-c\sqrt{\log x})\right)$$

for $q \le (\log x)^A$

Use lemma of Estermann:

②

<u>Lemma 17.1</u> (Estermann) Suppose that $f(s)$ is analytic for $|s-2| \leq 3/2$, and that $|f(s)| \leq M$ for s within this disc. Suppose that $F(s) = \zeta(s) f(s)$ can be written as a Dirichlet series for $\sigma > 1$, and define the coefficients $r(n)$ by means of the relation

$$F(s) = \sum_{n=1}^{\infty} r(n) n^{-s} \qquad (\sigma > 1).$$

Suppose in addition that $r(1) = 1$ and $r(n) \geq 0$ $(n \in \mathbb{N})$.

Then, whenever there exist $\sigma \in [19/20, 1)$ such that $f(\sigma) \geq 0$, one has

$$f(1) \geq \frac{1}{4}(1-\sigma) M^{-3(1-\sigma)}.$$

<u>Proof</u> (so far):

③

expand $$F(s) = \sum_{k=0}^{\infty} b_k (2-s)^k \qquad (|s-2|<1)$$

$$b_k = \frac{(-1)^k}{k!} F^{(k)}(2) = \frac{1}{k!} \sum_{n=1}^{\infty} r(n) n^{-2} (\log n)^k .$$

$$\downarrow$$

$$F(s) - \frac{f(1)}{s-1} = \sum_{k=0}^{\infty} (b_k - f(1))(2-s)^k$$

$|s-2|<1$

$\Downarrow$

$|s-2| \leq \frac{3}{2}$.

(17·1)

$$|b_k - f(1)| \leq 12M \left(2/3\right)^k$$

(Cauchy coefficient ineq. using $\left|F(s) - \frac{f(1)}{s-1}\right| \leq 12M$.)

$$\Downarrow$$

When $\frac{1}{2} \leq \sigma \leq 2$, have

$$\zeta(\sigma) f(\sigma) - \frac{f(1)}{\sigma - 1} \geqslant \sum_{k=0}^{K} (b_k - f(1))(2-\sigma)^k - 12M \sum_{k>K} \left(\frac{2}{3}(2-\sigma)\right)^k .$$

(17·2)

④ Observe that since $b_0 \geq 1$ and $b_k \geq 0$ $(k \in \mathbb{N})$,
and when $\frac{19}{20} \leq \sigma < 1$, have $\frac{2}{3}(2-\sigma) \leq \frac{7}{10}$, get

$$\zeta(\sigma) f(\sigma) - \frac{f(1)}{\sigma - 1} \geq 1 - f(1) \sum_{k=0}^{K} (2-\sigma)^k - \left(\frac{7}{10}\right)^{K+1} \cdot 12M \sum_{l=0}^{\infty} \left(\frac{7}{10}\right)^l$$

$$= 1 - f(1) \frac{1-(2-\sigma)^{K+1}}{1-(2-\sigma)} - \left(\frac{7}{10}\right)^{K+1} \frac{12M}{1-\frac{7}{10}}$$

$$\Rightarrow \quad 1 \leq -f(1) \cdot \frac{1-(2-\sigma)^{K+1}}{1-\sigma} + \zeta(\sigma) f(\sigma) - \frac{f(1)}{\sigma-1} + 40M \left(\frac{7}{10}\right)^{K+1}$$

$$\Rightarrow \quad 1 \leq f(1) \frac{(2-\sigma)^{K+1}}{1-\sigma} + \zeta(\sigma) f(\sigma) + 40M \left(\frac{7}{10}\right)^{K+1}$$

Choose

$$K = \left\lfloor \frac{\log(80M)}{\log(10/7)} \right\rfloor \longrightarrow \text{last term} \leq \frac{1}{2}$$

⑤ Also, since $\frac{19}{20} \le \sigma < 1$, have $\zeta(\sigma) > 0$,

and $f(\sigma) \geq 0$ by hypothesis. Thus

$$f(1)\,\frac{(2-\sigma)^{K+1}}{1-\sigma} \geq 1 - \frac{1}{2},$$

whence

$$\begin{aligned} f(1) &\geq \tfrac{1}{2}(1-\sigma)(2-\sigma)^{-K-1} \\ &\geq \tfrac{1}{2}(1-\sigma)(2-\sigma)^{-(\lfloor \log(80M) \rfloor / \log(\frac{10}{7}))+1} \\ &\geq \tfrac{10}{21}(1-\sigma)(80M)^{-\log(2-\sigma)/\log(10/7)} \\ &\geq \tfrac{10}{21} M^{-\log(2-\sigma)/\log(10/7)} \cdot (80)^{-\frac{\log \frac{21}{20}}{\log 10/7}} (1-\sigma) \\ &\geq \tfrac{1}{4} M^{-3(1-\sigma)} (1-\sigma), \end{aligned}$$

since $\log(10/7) \geq 1/3$ and $\log(2-\sigma) = \log(1 + 1-\sigma) \leq 1-\sigma$.

$$\Rightarrow \quad f(1) \geq \tfrac{1}{4}(1-\sigma)\, M^{-3(1-\sigma)}. \quad //$$

⑥ <u>Proof of Theorem 15.6.</u> Consider any $\varepsilon > 0$, say with $\varepsilon \leq 1/5$ (wlog). Observe that there is no loss of generality in supposing χ is primitive modulo q. For suppose χ is not primitive, so χ is induced by a primitive character χ^* modulo d with $d | q$, say $q = dr$. But then $\chi = \chi_0 \chi^*$ for principal character χ_0 mod q, and so

$$L(1,\chi) = L(1,\chi^*) \prod_{p|r} \left(1 - \frac{\chi^*(p)}{p}\right) \geq L(1,\chi^*) \frac{\varphi(r)}{r}$$

$$\underset{(\text{Siegel for } \chi^*)}{\gg} \quad C_1(\varepsilon) d^{-\varepsilon} \cdot C_2 r^{-\varepsilon} = C_1(\varepsilon) C_2(\varepsilon) (dr)^{-\varepsilon}$$

$$\geq C_1(\varepsilon) C_2(\varepsilon) q^{-\varepsilon}$$

(so by changing ε, get desired result). $\square$

(Recall: $\dfrac{\varphi(r)}{r} \gg \dfrac{1}{\log\log r}$)

⑦ Now can restrict to χ primitive (& quadratic) mod q.

(i) Suppose that there exists no quadratic character χ_1 such that $L(s,\chi_1)$ has a real zero $\beta_1 \in [1-\frac{\varepsilon}{4}, 1)$.

In this case we take $f(s) = L(s,\chi)$ and $\sigma = 1-\varepsilon/4$.

By Lemma 14.1, we have

$$\sup_{|s-2|\le 3/2} |L(s,\chi)| \ll \sup_{\substack{\frac{1}{2}\le\sigma\le\frac{7}{2}\\ |t|\le 3/2}} \left(1+(q\tau)^{1-\sigma}\right)\min\left\{\frac{1}{|\sigma-1|}, \log(q\tau)\right\}$$

$$\ll q^{1/2}.$$

$\zeta(s)\,L(s,\chi) = \sum_{n=1}^{\infty} r(n) n^{-s}$

with $r(n) = \sum_{d|n}\chi(d) \geq 0$ as in proof $L(1,\chi)>0$.

So can apply Lemma 17.1 with $M \ll q^{1/2}$. Thus

$$L(1,\chi) \geq \frac{1}{4}\left(\frac{\varepsilon}{4}\right) M^{-3\varepsilon/4} \gg \varepsilon\, q^{-3\varepsilon/8}.$$

Thus $L(1,\chi) > C_1(\varepsilon)\, q^{-\varepsilon}$ follows for suitable $C_1(\varepsilon)>0$. □

⑧ (ii) Suppose that there exists a quadratic primitive character χ_1 such that $L(s,\chi_1)$ has a real zero $\beta_1 \geq 1-\varepsilon/4$.

In this case, since $L(1,\chi_1) > 0$, there is a constant $c_2(\varepsilon)$ such that $L(1,\chi_1) \geq c_2(\varepsilon)\, q_1^{-\varepsilon}$ (just take $c_2(\varepsilon) = q_1^{\varepsilon} L(1,\chi_1)$). We now consider what happens for other primitive χ mod q with $\chi \neq \chi_1$.

We apply Lemma 17 i) with

$$f(s) = L(s,\chi)L(s,\chi_1)L(s,\chi\chi_1).$$

Check $f(s)\zeta(s)$ is a Dirichlet series (✓) with non-negative coefficients. To verify this, note

$$\log(\zeta(s)f(s)) = \sum_{n=1}^{\infty} \frac{\Lambda(n)}{\log n}\underbrace{(1+\chi(n)+\chi_1(n)+\chi\chi_1(n))}_{c(n)} n^{-s}$$

$$= \sum_{n=1}^{\infty} \frac{\Lambda(n)}{\log n}(1+\chi(n))(1+\chi_1(n))\, n^{-s}$$

⑨ has non-negative coefficients. Thus, on exponentiation, we get

$$\zeta(s) f(s) = \exp\left(\sum_{n=1}^{\infty} c(n) n^{-s} \right)$$

$$= \sum_{h=0}^{\infty} \left(\sum_{n=1}^{\infty} c(n) n^{-s} \right)^h \Big/ h! = \sum_{m=1}^{\infty} g(m) m^{-s}$$

non-negative coeffts. since $c(n) \geq 0$.

Observe also

$$\sup_{|s-2| \leq 3/2} \underset{\underset{L(s,\chi)L(s,\chi_1)L(s,\chi\chi_1)}{\shortparallel}}{|f(s)|} \ll q^{1/2} \cdot q_1^{1/2} \cdot (q q_1)^{1/2} \ll q q_1 .$$

Thus, we can apply Lemma 17.1 with $M = C_3 q_1 q$, say, and $\sigma = \beta_1$. Thus

$$
\begin{aligned}
f(1) &\geqslant \tfrac{1}{4}\left(c_3 q_1 q\right)^{-3(1-\beta_1)} \\
&\geqslant \tfrac{1}{4}\left(c_3 q q_1\right)^{-3\varepsilon/4}.
\end{aligned}
$$

But also from Lemma 14·1,

$$f(1) = L(1,\chi)L(1,\chi_1)L(1,\chi\chi_1) \ll L(1,\chi)(\log q_1)(\log qq_1),$$

Whence

$$L(1,\chi)\left(\log(qq_1)\right)^2 \gg (qq_1)^{-3\varepsilon/4}$$

$$\Downarrow$$

$$L(1,\chi) \gg_{q_1,\varepsilon} q^{-3\varepsilon/4}/(\log q)^2 \gg q^{-\varepsilon}.$$

Thus, there exists $C_1(\varepsilon) > 0$ s.t. $L(1,\chi) \gg q^{-\varepsilon}$. //